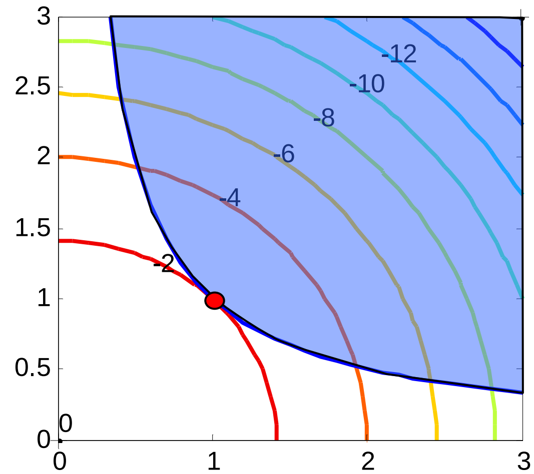
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Problem Set 2 – Optimality Conditions, GAMS
Solutions

See: PS06_2.gms, PS06_2.lst

1. Minimize: $x_1^2 + x_2^2$
   such that: $x_1 x_2 \geq 1$
   $x_1, x_2 \geq 0$



Candidate solution: (1, 1)

$n = 2, m_A^* = 1, \rho^* = 1, n - \rho^* = 1$

i) Feasible

ii), iv) Stationarity, Lagrange

$$\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$\lambda_1 = 2 > 0$

iii) $L = -x_1^2 - x_2^2 - \lambda_1 [-x_1 x_2 - 1]$

$$\frac{\partial L}{\partial x_i \partial x_j} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \quad \frac{\partial g_i}{\partial x_j} Z_{jk} = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ -a \end{bmatrix} = 0$$

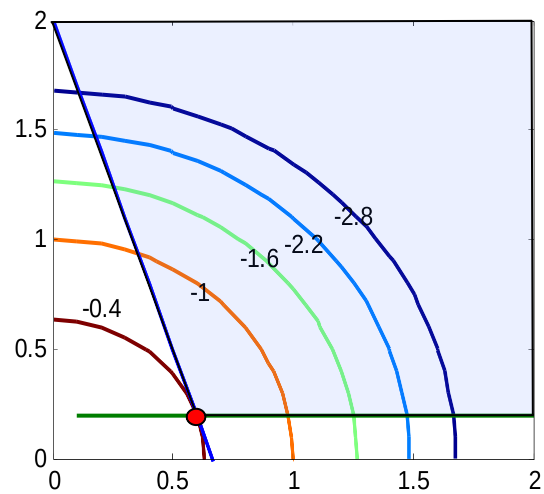
$$W_{kl} = Z_{ki} \frac{\partial L}{\partial x_i \partial x_j} Z_{lj} = -8a^2 \leq 0$$

All necessary conditions are satisfied.
Feasible region convex, local max. is global max.

2. Minimize: $x_1^2 + x_2^2$
   such that: $x_2 \geq 2 - 3x_1$
   $x_2 \geq 1/5$
   $x_1, x_2 \geq 0$

Candidate solution: (3/5, 1/5) = (0.6, 0.2)

$n = 2, m_A^* = 2, \rho^* = 2, n - \rho^* = 0$



i) Feasible

ii), iv) Stationarity, Lagrange

$$\begin{bmatrix} -6/5 \\ -2/5 \end{bmatrix} = \lambda_1 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -3/5 \end{bmatrix}$$

$\lambda_1 = 2/5 > 0$, $\lambda_2 = 0 \geq 0$

Problem is not degenerate but constraint 2 is redundant (solution would not change if it is omitted, since its Lagrange multiplier is 0)

iii) Curvature not applicable since $n - \rho^* = 0$

All necessary conditions are satisfied.
Feasible region convex, local max. is global max.

3. Minimize: $(x_1\text{-}1)^2 + 2x_2^2 + x_3^2$

such that: $x_3 = 2\, x_1^{1/2}$

$x_2 + 2\, x_1 \geq 2$

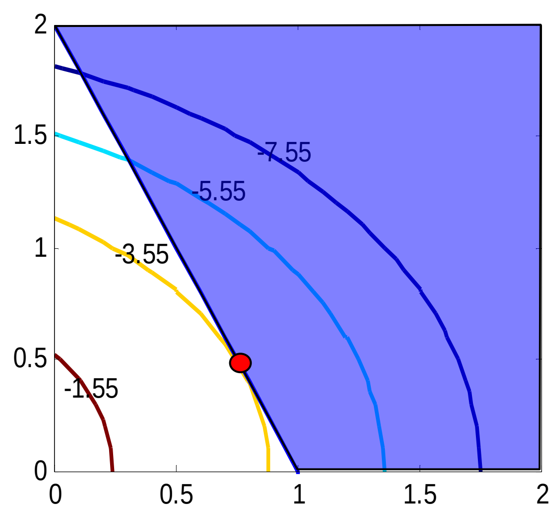
$x_1, x_2 \geq 0$

Substitute equality constraint into objective to reduce problem to 2 unknowns.

Minimize: $(x_1+1)^2 + 2x_2^2$

$x_2 + 2\, x_1 \geq 2$

$x_1, x_2 \geq 0$



Candidate solution: (7/9, 4/9) = (0.778, 0.444)

$n = 2, m_A^* = 1, \rho^* = 1, n - \rho^* = 1$

i) Feasible

ii), iv) Stationarity, Lagrange

$$\begin{bmatrix} -32/9 \\ -16/9 \end{bmatrix} = \lambda_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$\lambda_1 = 16/9 > 0$

iii) $L = (x_1+1)^2 + 2x_2^2 + \lambda[-x_2 - 2x_1 - 2]$

$$\frac{\partial L}{\partial x_i \partial x_j} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \quad \frac{\partial g_i}{\partial x_j} Z_{jk} = \begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ -2a \end{bmatrix} = 0$$

$$W_{kl} = Z_{ki} \frac{\partial L}{\partial x_i \partial x_j} Z_{lj} = -18a^2 \le 0$$

All necessary conditions are satisfied.
Feasible region convex, local max. is global max.

4. Maximize: $x_1 + 2x_2$
   such that: $x_1 + x_2 \le 1$
   $2x_2 \le x_1 + 2$
   $x_1, x_2 \ge 0$

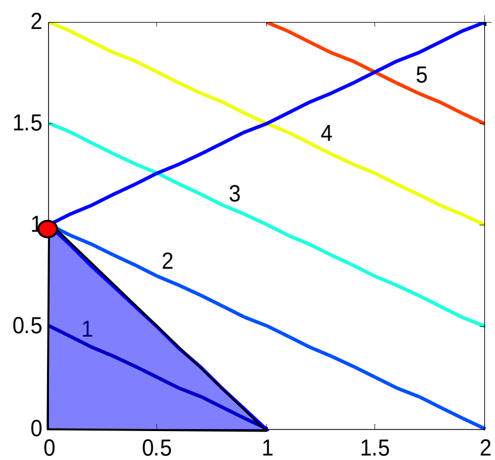
Candidate solution: (0,1)

$n = 2, m_A^* = 3, \rho^* = 2, n - \rho^* = 0$

i) Feasible
ii), iv) Stationarity, Lagrange

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Linearly-dependent constraint gradients
Degenerate problem
Non-unique Lagrange multipliers
iii) Curvature condition always satisfied for LP problems



Problem becomes non-degenerate and well-posed if we delete constraint 2 ($\lambda_2 = 0$) so:

$\lambda_1 = 2 > 0, \lambda_3 = 1 > 0$

Then all necessary conditions are satisfied.
Feasible region convex, local max. is global max.

5. Maximize: $3x_1 + 7x_2$
   such that: $x_1 - x_2 \ge 0$
   $x_1 + x_2 \le 7/2$
   where x_1 and x_2 are nonnegative integers

Feasible integer solutions: (0,0), (1,0), (1,1), (2,0), (2,1), (3,0)
By enumeration, global maximum is (2,1)
Necessary conditions ii)-iv) do not apply