10.001: System of Linear Equations, Part 1.

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1 Introduction

This lecture as well as the following one on Tuesday will address certain mathematical and computational issues related to the solution of a system of linear equations. We will first review the mathematical aspects of the problem which form the basis of *linear algebra*. The mathematical concepts will then be utilized to understand algorithms which will enable us to solve systems of linear equations, a problem which is central to the subject of numerical analysis.

2 What are linear equations?

A linear equation can be mathematically represented as

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = b, \tag{1}$$

where $x_1, x_2, \cdots, x_n$ are the n *unknowns* in the equation. The numbers (constants) $c_1, c_2, \cdots, c_n$ are the *coefficients* of the unknowns. Eq. 1 represents a condition to be satisfied by the n unknowns. It is called a *linear* equation because it contains only the first powers of the unknowns (i.e., no term such as x_1^2 or x_i^p where $p \neq 1$ appears in Eq. 1).

Examples of linear equations: $10x = 5$; $7x + 4y = 17$, $x + y/23.486 - 11.786z = 4.579$ etc.

3 System of linear equations

In general, a system of m linear equations in n unknowns may be written as follows.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$
\begin{aligned}
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots &= \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned} \tag{2}
$$

Evidently, the representation of a large system of equations in the form used in Eq. 2 is cumbersome and unwieldy. Hence, we would like to express these equations compactly. The most natural way to do this is to use *matrices* and *vectors*. We will first go through the basic concepts of matrices and vectors and come back to Eq. 2 later.

4 Vectors: Preliminaries

We are familiar with vector quantities, we deal with them all the time, such as forces, velocities and displacements. We are also familiar with their geometric representation in an orthogonal Cartesian coordinate system in 1, 2, or 3-dimensional space as arrows. For instance, when we represent a force vector $\mathbf{F}$ in this fashion, the length of the arrow is proportional to the magnitude of the force and its direction represents the direction in which the force is applied[1]. We can find the projection of a 2-dimensional vector $\mathbf{F}$ onto the mutually orthogonal x and y axes of a Cartesian coordinate system, say F_x and F_y respectively and write:

$$\mathbf{F} = F_x \mathbf{e_x} + F_y \mathbf{e_y}, \tag{3}$$

where $\mathbf{e_x}$ and $\mathbf{e_y}$ are the *unit vectors*, i.e., vectors of unit length along the x and y axes respectively. So, if we keep in mind that our basis is the coordinate system represented by the $x - y$ plane, then with repect to that basis, the vector $\mathbf{F}$ is simply a collection of two numbers, (F_x, F_y).

We would like to extend these familiar ideas to n dimensions (where n is finite) so that we could bring under our formulation the problems related to systems of linear equations. We hence consider any $n$tuple, such as $(x_1, x_2, ..., x_n)$, as an n-dimensional vector, say $\mathbf{x}$. Physically, such an $n$tuple could represent the mole fractions of various components in a petrochemical feed, the current passing through different parts of an electrical network, or the temperature at different radial locations of a tubular flow reactor etc. We represent the vector $\mathbf{x}$ algebraically as a single column consisting of the n numbers (elements) $(x_1, x_2, ..., x_n)$ as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{4}$$

[1]We use bold face to denote vectors and matrices.

Similarly, we use the notation $\mathbf{x}^T$ to denote the *transpose* of the *column vector* $\mathbf{x}$ defined by Eq. 4. $\mathbf{x}^T$ is simply given by the *row vector*

$$\mathbf{x}^T = (x_1 \, x_2 \, \cdots \, x_n) \, . \tag{5}$$

We also introduce the compact notation $\mathbf{x} = \{x_i\}$ to denote a vector where the curly brackets are intended to remind us that we are dealing with an n dimensional vector with components x_i, $i = 1, \cdots, n$. We now proceed to define certain vector operations.

4.1 Vector Operations

1. **Addition**: If $\mathbf{x} = \{x_i\}$ and $\mathbf{y} = \{y_i\}$ are two n dimensional vectors, then their addition results in another n dimensional vector, say $\mathbf{z}$ where $\mathbf{z} = \{x_i + y_i\}$.
2. **Multiplication by a scalar**: If $\mathbf{x} = \{x_i\}$ and α a scalar then $\alpha\mathbf{x} = \{\alpha x_i\}$.

In addition to the 2 operations listed above, I would like to introduce the concept of the *dot product* of two n dimensional vectors, say $\mathbf{x}$ and $\mathbf{y}$. The dot product, denoted by $\mathbf{x}.\mathbf{y}$ (pronounced x dot y), results in a scalar (a real number in the case of real vectors), according to the equation:

$$\mathbf{x}.\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{6}$$

The dot product has a geometric meaning, as the length of $\mathbf{x}$ times projection of vector $\mathbf{y}$ onto vector $\mathbf{x}$. Hence, the dot product of two orthogonal (i.e., mutually perpendicular) vectors is 0. Also notice that $\mathbf{x}.\mathbf{x}$ is equal to the square of the length of $\mathbf{x}$.

We should also be familiar with the algebraic representation of a unit vector, i.e., a vector of unit length. We use the symbol $\mathbf{e}^i$ to denote the unit vector in the ith direction, and this vector consists of 0 for all its components except for the ith one which is unity. For instance, $\mathbf{e}^1 = (1, 0, 0, \cdots, 0)^T$ and $\mathbf{e}^2 = (0, 1, 0, \cdots, 0)^T$ etc. It is now evident that any vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} . \tag{7}$$

Eq. 7 expresses an arbitrary n-dimensional vector $\mathbf{x}$ as a *linear combination* of the n unit vectors. This is possible because the set of n unit vectors form a *linearly independent* set of vectors, i.e., none of the n unit vectors in Eq. 7 can be expressed as a linear combination of any one of the remaining $n - 1$ unit vectors. In general, we have the following definition:

Definition (Linear Independence): We say that the k vectors $\mathbf{x}^1$, $\mathbf{x}^2$ ...$\mathbf{x}^k$ are linearly independent (LI) if the condition $\alpha_1\mathbf{x}^1 + \alpha_2\mathbf{x}^2 + \cdots + \alpha_k\mathbf{x}^k = \mathbf{0}$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Examples:
1. $\mathbf{x}^1 = (1, 2, 4)$ and $\mathbf{x}^2 = (2, 5, 3)$ are LI because to satisfy $\alpha_1\mathbf{x}^1 + \alpha_2\mathbf{x}^2 = 0$, we need to satisfy the three equations: $\alpha_1 + 2\alpha_2 = 0$; $2\alpha_1 + 5\alpha_2 = 0$ and $4\alpha_1 + 3\alpha_2 = 0$, which is possible only if $\alpha_1 = \alpha_2 = 0$.
2. $\mathbf{x}^1 = (1, 2, 4)$ and $\mathbf{x}^2 = (2, 4, 8)$ are not LI. Why?
3. Any n-dimensional vector can be expressed as a linear combination of a set of n LI vectors (Test this for $n = 3$). The set of n LI vectors you choose defines a *basis* for the n-dimensional vector space. If we choose the unit vectors along the n mutually orthogonal (perpendicular) directions (see Eq. 7), then it is called the *natural basis*.
4. Using Eq. 7 and the definition of the dot product, deduce that $x_i = \mathbf{x}.\mathbf{e}^i$ for $i = 1, 2, \cdots, n$. Also deduce that $\mathbf{e}^i.\mathbf{e}^j$ is 1 if $i = j$ and 0 otherwise.

5 Matrices: Preliminaries

The simplest way to define an $m \times n$ (pronounced m by n) matrix is as an array with mn entries (elements) arranged into m rows and n columns. We will also use a curly bracket notation to denote matrices, i.e., the matrix $\mathbf{A}$ can be written as $\mathbf{A} = \{a_{ij}\}$ for $i = 1, 2, ..., m$ and $j = 1, 2, .., n$. In the expanded form, we can represent this as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{8}$$

From Eq. 8, we can see that the rectangular matrix $\mathbf{A}$ may be thought of as a collection of n column vectors of dimension m or as the collection of m row vectors of dimension n (Identify these vectors). Similarly, we could think of an n-dimensional column vector defined by Eq. 4 as an $n \times 1$ matrix or the n-dimensional row vector of Eq. 5 as a $1 \times n$ matrix. An important class of matrices arise when the number of rows are equal to the number of columns, i.e., $m = n$. Such a matrix is called a *square matrix* of dimension n.

5.1 Matrix Operations/Properties

1. **Addition**: $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ be two $m \times n$ matrices. Then the $(m \times n)$ matrix $\mathbf{C}$ which results from the addition of $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{C} = \{c_{ij}\} = \{a_{ij} + b_{ij}\}$. For instance, if

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 9 & 14 & 5 & -2 \\ 8 & 3 & 7 & -32 \end{pmatrix} \tag{9}$$

and

$$\mathbf{B} = \begin{pmatrix} -3 & 8 & -5 & 1 \\ 0 & 4 & 5 & 5 \\ 2 & 6 & 17 & 2 \end{pmatrix}, \tag{10}$$

then the matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is given by

$$\mathbf{C} = \begin{pmatrix} 0 & 10 & -4 & 1 \\ 9 & 18 & 10 & 3 \\ 10 & 9 & 24 & -30 \end{pmatrix}. \tag{11}$$

2. **Transposition**: Let $\mathbf{A} = \{a_{ij}\}$ be an $m \times n$ matrix. Then the transpose of $\mathbf{A}$, denoted by $\mathbf{A}^T$, is the $n \times m$ matrix, $\{a_{ji}\}$. i.e., the transpose is obtained by interchanging the rows and columns of the original matrix. For instance, the transpose of the matrix,

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 8 & 3 & 7 \end{pmatrix} \tag{12}$$

is given by

$$\mathbf{A}^T = \begin{pmatrix} 3 & 8 \\ 2 & 3 \\ 1 & 7 \end{pmatrix}. \tag{13}$$

A square matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A} = \mathbf{A}^T$. It is *skew symmetric* if $\mathbf{A} = -\mathbf{A}^T$. For instance, the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 6 & 7 \\ 1 & 7 & 5 \end{pmatrix} \tag{14}$$

is symmetric.

3. **Matrix Multiplication**: Let $\mathbf{A} = \{a_{ij}\}$ be an $m \times n$ matrix and let $\mathbf{B} = \{b_{ij}\}$ be an $n \times p$ matrix. Then the matrix product, $\mathbf{A}.\mathbf{B}$ (pronounced A dot B or simply AB) is an $m \times p$ matrix, say $\mathbf{C} = \{c_{ij}\}$, where each element c_{ij} is computed as

$$c_{ij} = \sum_{k=1,n} a_{ik} b_{kj} \text{ for } i = 1, \cdots m;\ j = 1, \cdots p. \tag{15}$$

Notice that multiplication operation is defined only if the number of columns of matrix **A** is equal to the number of rows of matrix B (n in the case above). The operation defined by Eq. 15 is often referred to as the 'row times column' operation, i.e., to get the element c_{ij} of the product matrix **C**, we form the dot product of the ith row vector of matrix **A** with the jth column vector of matrix **B**.
Example: Let **A** be given by the 2×3 matrix of Eq. 12. Let **B** be given by the 3×2 matrix

$$\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 5 & 1 \\ 0 & 6 \end{pmatrix}. \tag{16}$$

Then the product $\mathbf{C} \equiv \mathbf{A}.\mathbf{B}$ is the 2×2 matrix given by

$$\mathbf{C} = \begin{pmatrix} 3 & 2 & 1 \\ 8 & 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 5 & 1 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 3 \times 4 + 2 \times 5 + 1 \times 0 = 22 & 3 \times 0 + 2 \times 1 + 1 \times 6 = 8 \\ 8 \times 4 + 5 \times 3 + 7 \times 0 = 47 & 8 \times 0 + 3 \times 1 + 7 \times 6 = 45 \end{pmatrix}. \tag{17}$$

In a similar fashion, verify that the matrix $\mathbf{D} \equiv \mathbf{B}.\mathbf{A}$ is the 3×3 matrix

$$\mathbf{D} = \begin{pmatrix} 12 & 8 & 4 \\ 23 & 13 & 12 \\ 48 & 18 & 42 \end{pmatrix}. \tag{18}$$

In general, $\mathbf{A}.\mathbf{B} \neq \mathbf{B}.\mathbf{A}$, i.e., matrix multiplication is *non-commutative.*

4. **Diagonal and Unit (Identity) Matrices**: A square matrix is diagonal if all its entries except the diagonal ones are zeroes. If each of the diagonal entries of a diagonal matrix is unity, the matrix is called a unit matrix. For instance, the 3×3 matrix

$$\mathbf{D} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 42 \end{pmatrix} \tag{19}$$

is diagonal. Similarly, the 3×3 unit matrix, denoted by $\mathbf{I}_3$ is given by (the subscript 3 denotes the dimension of the matrix)

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

It can now be verified that multiplying any 3×3 matrix, say **A**, by $\mathbf{I}_3$ results in **A** itself.

5. **Upper and Lower Triangular Matrices**: A square matrix is upper triangular if each of the entries below its diagonal is zero. Simlarly, a lower triangular matrix has each of its entries above its diagonal as zero.

6 Representation of System of Linear Equations using Matrices

We now go back to Eq. 2 and see how the concepts presented in the previous sections can be applied to yield a compact representation of the system of equations using matrices. We define **A** to be $m \times n$ vector of coefficients, or simply the *coefficient matrix* as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{21}$$

Now, we define **x** to be the n-dimensional column vector of unknowns or equivalently as a $n \times 1$ matrix as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{22}$$

Evidently, the product **A**.**x** will result in an $m \times 1$ matrix (i.e., simply a m-dimensional column vector), given by

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \end{pmatrix}, \tag{23}$$

which, according to Eq. 2, is simply the right hand side vector, **b**. So, we can now express the system of linear equations simply as:

$$\mathbf{A}.\mathbf{x} = \mathbf{b}. \tag{24}$$

Our task is to find the solution vector **x**, given the coefficient matrix **A** and the *load* vector **b**. We will restrict our further discussions to systems of n equations in as many unknowns, i.e., the coefficient matrix a will be a $n \times n$ square matrix.

The basic idea behind the algorithms is to reduce Eq. 24 to a simpler form amenable for solution. The simplest of such forms involves the reduction of **A** to a diagonal matrix, i.e., after the reduction process, we will end up with (in the $n \times n$ case):

$$\begin{pmatrix} \alpha_{11} & 0 & \cdots & 0 \\ 0 & \alpha_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_{nn} \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \tag{25}$$

where the solution is directly given by $x_i = \beta_i/\alpha_i$ (we will assume for the present that $\alpha_i \neq 0$).

Prior to discussing algorithms to achieve reduction to a diagonal or a triangular form, let's review in some detail the *existence* and *uniqueness* of solutions to Eq. 24 ($m = n$ case) and *invariant operations*.

6.1 Existence/Uniqueness of solution

The general existence/uniqueness theorem for a system of linear equations (called the Fredholm's alternative) is one of the fundamental theorems of linear algebra. However, we refrain from a discussion of that theorem, and simply limit our discussion to the special case of the $n \times n$ system. Our aim is to identify a *solvability condition* for the system of equations $\mathbf{A}.\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ matrices or simply, n-dimensional vectors. In other words, we would like to identify a criterion the satisfaction of which will guarantee us that there exists *exactly one* solution to the system of equations $\mathbf{A}.\mathbf{x} = \mathbf{b}$.

We will do this by considering an example where $n = 2$. Consider the system of equations given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} . \tag{26}$$

The solution to this equation can be written as:

$$\begin{aligned} x_1 &= \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_2 &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}. \end{aligned} \tag{27}$$

We notice from Eq. 27 that the denominators of the expressions for both x_1 and x_2 are the same, namely, $a_{11}a_{22} - a_{12}a_{21}$. Evidently, if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, we have exactly one solution, as given by Eq. 27. The number $a_{11}a_{22} - a_{12}a_{21}$ is referred to as the *determinant* of the coefficient matrix in Eq. 26 and the condition for a unique solution to exist is then given by the requirement that the coefficient matrix should have a non-zero determinant. We will not enter into the definition of the determinant for a general $n \times n$ matrix, we rather will try to identify certain structure in the matrix which causes the determinant to vanish or equivalently, the conditions under which a matrix becomes *singular*.

Let's examine the condition $a_{11}a_{22} - a_{12}a_{21} = 0$. This implies that $a_{21}/a_{11} = a_{22}/a_{12}$, say $= r$. i.e., the elements of row 2 of the coefficient matrix are simply multiples of the corresponding elements of row 1. (or equivalently, column 2 is a multiple of column 1). Based on the definition of *linear independence* of vectors developed in section 4.1, we say that the vectors formed by the 2 rows of the coefficient matrix are not linearly independent. In general, if any one of the rows (columns) of an $n \times n$ matrix can be expressed as a linear combination of the a set of rows (columns) from remaining of the remaining rows (columns), then the we do not have a linearly independent set of n row vectors in the coefficient matrix, as a result, the system of equations will not have a *unique* solution.

Examples:
Example E_1: The system of equations, given by

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \\ 20 \end{pmatrix} \tag{28}$$

has the unique solution $x = 1, y = 2, z = 3$ (How will we prove this?).
Example E_2: The system of equations, given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \tag{29}$$

does not have a solution at all (notice that in the coefficient matrix, row 1 = row 2 - row 3).
Example E_3: The system of equations, given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \tag{30}$$

(notice that the only change compared to example E_2 is in the right hand side vector) has no unique solution, but it has *infinitely many* solutions, given by $x = a, y = 2, z = -(1 + a)$ as the real constant a can take infinitely many values and still satisfy the system of equations of example E_3.

The generalization of the above observation to $n \times n$ systems is called the *Fredholm's alternative*, a discussion of which may be found in any standard textbook on linear algebra. For the purpose of our course, we notice that: EITHER (1). the coefficient matrix has n linearly independent rows (columns) which implies that the $n \times n$ system has exactly one solution (Example E_1) OR (2). if one or more of the rows (columns) of the coefficient matrix is linearly dependent on a set of rows (columns) of the remaining ones which implies either (a). the system of equations has no solution at all (Example E_2) or (b). the system of equations has infinitely many solutions (Example E_3). To determine whether situation (a) or (b) occurs when case (2) condition is satisfied requires a more detailed analysis of the *homogeneous* system of equations $\mathbf{A}^T.\mathbf{y} = \mathbf{0}$, and will not be discussed here. We focus our attention on the Case (1) scenario, i.e., the case for which the coefficient matrix is non-singular. For any non-singular $n \times n$ matrix $\mathbf{A}$, we can always define the inverse matrix denoted by $\mathbf{A}^{-1}$, such that

$$\mathbf{A}.\mathbf{A}^{-1} = \mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}_n, \tag{31}$$

where $\mathbf{I}_n$ is the n-dimensional identity matrix. So, if we premultiply the matrix equation $\mathbf{A}.\mathbf{x} = \mathbf{b}$ with $\mathbf{A}^{-1}$, we get $\mathbf{I}_n.\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, which essentially solves the problem. However, finding $\mathbf{A}^{-1}$ is as difficult as solving the system of equations. Most algorithms which are designed for solving $\mathbf{A}.\mathbf{x} = \mathbf{b}$ can also be used to find the inverse of the coefficient matrix.