10.301 Problem Set #3

Solutions

4.6 Show that the streamlines for a flow whose velocity components are $u = c(x^2 - y^2)$ and $v = -2cxy$, where c is a constant, are given by the equation $x^2 y - y^3/3 = \text{constant}$. At which point (points) is the flow parallel to the y-axis? At which point (points) is the fluid stationary?

4.6 Solution

A streamline is a line that is everywhere tangent to the velocity field. In two-dimensions:

Along the streamline:

$dx = u\,dt \qquad dy = v\,dt$

$$dt = \frac{dx}{u} = \frac{dy}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v}{u}$$

Plugging in u and v:

$$\frac{dy}{dx} = \frac{-2cxy}{c(x^2-y^2)} = \frac{-2xy}{(x^2-y^2)} \qquad (1)$$

Now take a look back at the equation given in the problem

$$x^2 y - y^3/3 = \text{constant} \qquad (2)$$

Differentiate this with respect to x. Remember to use the chain rule.

$$x^2 \frac{dy}{dx} + 2xy - y^2 \frac{dy}{dx} = 0 \qquad (3)$$

Rearranging,

$$\frac{dy}{dx} = \frac{-2xy}{(x^2-y^2)} \quad \checkmark \qquad (4)$$

This matches the result in (1).

The flow will be parallel to the y-axis when $u = 0$ (no component of the velocity in the x-direction).

$$u = c(x^2-y^2) = 0$$

$$x^2 = y^2$$

$$\boxed{y = \pm x}$$

Another method: When flow is parallel to the y-axis, $\frac{dy}{dx} = \infty$

$$\frac{dy}{dx} = \frac{-2xy}{(x^2-y^2)}$$

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$\frac{dy}{dx} = \infty$ when the denominator $= 0$

$x^2 - y^2 = 0 \Rightarrow y = \pm x$

The flow will be stationary when $u = 0$ and $v = 0$ (when the velocity components of the 2D velocity field are equal to zero).

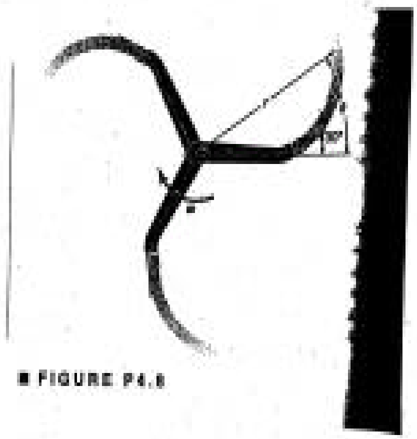
$u = x^2 - y^2 = 0$ and $v = -2xy = 0$

$y = \pm x$ $\quad\quad xy = 0$

The only point that satisfies both conditions is located at the origin $(0, 0)$.

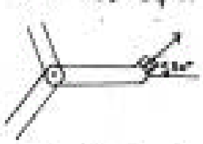
4.8

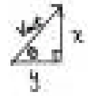
4.8 Water flows from a rotating lawn sprinkler as shown in Video V4.6 and Figure P4.8. The end of the sprinkler arm moves with a speed of ωR, where $\omega = 10$ rad/s is the angular velocity of the sprinkler arm and $R = 0.5$ ft is its radius. The water exits the nozzle with a speed of $V = 10$ ft/s relative to the rotating arm. Gravity and the interaction between the air and the water are negligible. (a) Show that the pathlines for this flow are straight radial lines. *Hint:* Consider the direction of flow (relative to the stationary ground) as the water leaves the sprinkler arm. (b) Show that at any given instant the stream of water that came from the sprinkler forms an arc given by $r = R + (V_r/\omega)\theta$, where the angle θ is as indicated in the figure and V_r is the water speed relative to the ground. Plot this curve for the data given.



■ FIGURE P4.8

(a) A pathline of a particle indicates the position of a particle in a fluid as it flows from one point to the next.
Before exiting the sprinkler, a water particle travels in an outward radial direction. Once it leaves the arm, the water particle no longer experiences the centripetal acceleration (due to inertia) and one can expect the water particle to travel in a straight line.





$R = 0.5$ ft
$\omega = 10$ rad/s
$V_{out} = 10$ ft/s
$\theta = 30°$

The velocity of a fluid particle right as it leaves the arm is:

$\underline{V} = -R\omega \underline{j} + V_{out}\cos\theta\, \underline{i} + V_{out}\sin\theta\, \underline{j}$

$\underline{V} = \begin{Bmatrix}\text{velocity component due} \\ \text{to the sprinkler arm}\end{Bmatrix} + \begin{Bmatrix}\text{velocity component} \\ \text{in the x-direction}\end{Bmatrix} + \begin{Bmatrix}\text{velocity component} \\ \text{in the y-direction}\end{Bmatrix}$

Thus:

$u = V_{out}\cos\theta$

$v = -R\omega + V_{out}\sin\theta$

the pathline of a particle (the location of the particle as a function of time) can be obtained from the velocity field and the definition of velocity.

$$\frac{dx}{dt} = u = V_{out} \cos\theta \quad \text{and} \quad \frac{dy}{dt} = v = -R\omega + V_{out}\sin\theta$$

V_{out}, θ, R, & ω are all constants, thus we can integrate

$$x = V_{out}(\cos\theta)\,t + C_1$$

$$y = (-R\omega + V_{out}\sin\theta)\,t + C_2$$

At $t = 0$, the position of the fluid particle is $x = 0$, $y = 0$. Using this initial condition to solve for C_1 and C_2:

$$0 = 0 + C_1$$

$$0 = 0 + C_2 \quad \Rightarrow \quad C_1 = C_2 = 0$$

$$x = V_{out}(\cos\theta)\,t$$

$$y = (-R\omega + V_{out}\sin\theta)\,t$$

Plugging in values, noting again that V_{out}, θ, R & ω are constant:

$$x = (10\ \text{ft/s})(\cos 30^\circ)\,t = (10\ \text{ft/s})\left(\frac{\sqrt{3}}{2}\right)t$$

$$\boxed{x = (5\sqrt{3}\ \text{ft/s})\,t}$$

$$y = \left[(-0.5\ \text{ft})\left(10\ \frac{\text{rad}}{\text{s}}\right) + (10\ \text{ft/s})\left(\frac{1}{2}\right)\right]t$$

$$\boxed{y = 0}$$

Thus, the pathlines for this flow are straight radial lines.

) Show that at any given instant the stream of water that came from the sprinkler forms an arc given by $r = R + (V_a/\omega)\theta$.

V_a = water speed relative to the ground

First, expand the derivative $\frac{dx}{dt}$ using the chain rule

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} \qquad (1)$$

Remember that $\frac{dx}{dt}$ also equals u

$$\frac{dx}{dt} = u = V_{out}\cos\theta = V_a \quad \text{(velocity from a fixed ground frame of reference)} \qquad (2)$$

From the diagram,

$$x = r\cos\theta \qquad (3)$$

Also, we know that

$$\frac{d\theta}{dt} = \omega \qquad (4)$$

Substituting in expressions into (1):

$$\frac{dx}{dt} = V_a = \cos\theta \frac{dr}{dt} - r\sin\theta\,\omega \qquad (5)$$

We also know that

$$\theta = \omega t \qquad (6)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \qquad (7)$$

Thus (5) becomes

$$V_a = \cos\theta \frac{dr}{d\theta}\omega - \omega r\sin\theta \qquad (8)$$

Dividing by ω

$$\frac{V_a}{\omega} = \cos\theta \frac{dr}{d\theta} - r\sin\theta \qquad (9)$$

Using the initial conditions:

@ $t = 0$, $\theta = 0$ and $r = R$

One could solve this ordinary differential equation.

OR, one can make the simplifying assumption that for short times $\omega t \ll 1$, θ is small and $\cos\theta = 1$ and $\sin\theta = 0$

Therefore (9) becomes

$$\frac{V_a}{\omega} = \frac{dr}{d\theta} \qquad (10)$$

Integrating,

$$r = \frac{V_a}{\omega}\theta + C \qquad (11)$$

At $r = R$, $\theta = 0$ therefore

$$R = 0 + C \Rightarrow C = R \qquad (12)$$

Substituting C in (11)

$$\boxed{r = \frac{V_a}{\omega}\theta + R}$$ ✓

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Alternate solution :

$$\frac{dr}{V_r} = \frac{d\theta}{V_\theta}$$

$$V_\theta\, dr = V_r\, dr$$

$$V_\theta = -\omega$$

$$V_r = V_a$$

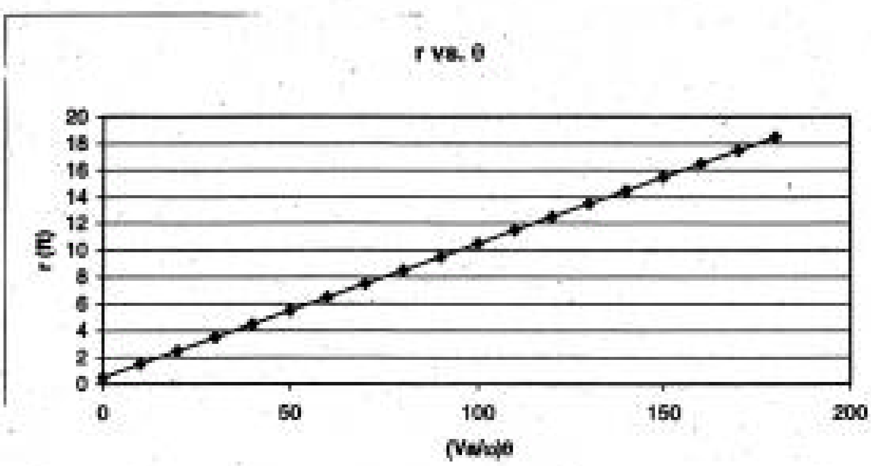
$$\frac{dr}{V_a} = \frac{d\theta}{-\omega}$$

$$\int_r^R -\omega\, dr = \int_0^\theta V_a\, d\theta$$

$$-\omega(R-r) = V_a \theta$$

$$R - r = -\frac{V_a \theta}{\omega}$$

$$r = \frac{V_a \theta}{\omega} + R \quad ✓$$



Radius r is linearly dependent on θ.

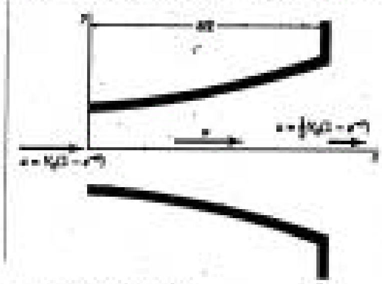
In the above graph : θ is in degrees

ω = 10 rad/s

R = 0.5 ft

V_a = 1 ft/s (arbitrarily chosen)

4.13 **4.23** As a valve is opened, water flows through the diffuser shown in Fig. P4.23 at an increasing flowrate so that the velocity along the centerline is given by $\mathbf{V} = u\hat{\mathbf{i}} = V_0(1 - e^{-ct})(1 - x/\ell)\hat{\mathbf{i}}$, where u_0, c, and ℓ are constants. Determine the acceleration as a function of x and t. If $V_0 = 10$ ft/s and $\ell = 5$ ft, what value of c (other than $c = 0$) is needed to make the acceleration zero for any x at $t = 1$ s? Explain how the acceleration can be zero if the flowrate is increasing with time.



■ FIGURE P4.23

Along the centerline: $\underline{V} = V_0(1-e^{-ct})(1-x/\ell)\,\hat{i}$

To find acceleration, take the material derivative of velocity

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + (\underline{V}\cdot\underline{\nabla})\underline{V}$$

Along the centerline, there is only an x-component of velocity, thus, the x-component of acceleration is

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad u = V_0(1-e^{-ct})(1-x/\ell)$$

First, find $\partial u/\partial t$ and $\partial u/\partial x$

$$\frac{\partial u}{\partial t} = V_0(1-x/\ell)(-e^{-ct})(-c) = cV_0(e^{-ct})(1-x/\ell)$$

$$\frac{\partial u}{\partial x} = V_0(1-e^{-ct})\left(-\tfrac{1}{\ell}\right) = -\frac{V_0}{\ell}(1-e^{-ct})$$

Substituting into the expression for a_x:

$$a_x = cV_0(e^{-ct})(1-x/\ell) + V_0(1-e^{-ct})(1-x/\ell)\left[\left(-\frac{V_0}{\ell}\right)(1-e^{-ct})\right]$$

$$a_x = V_0(1-x/\ell)\left[c(e^{-ct}) - (1-e^{-ct})^2(V_0/\ell)\right]$$

Setting $a_x = 0$:

$$0 = V_0(1-x/\ell)\left[c(e^{-ct}) - (1-e^{-ct})^2(V_0/\ell)\right]$$

$$0 = c(e^{-ct}) - (1-e^{-ct})^2(V_0/\ell)$$

Plugging in values for V_0, ℓ, and t

$$0 = c(e^{-c}) - (1-e^{-c})^2(10/5)$$

$$0 = c(e^{-c}) - 2(1-e^{-c})^2$$

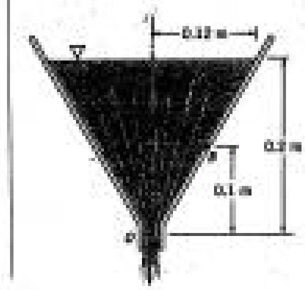
Solving numerically,

$$c = 0.490$$

b) The acceleration can still be zero if the flowrate is increasing with time because the cross sectional area of the valve is also increasing. Acceleration depends not only on the change of velocity with respect to time ($\partial u/\partial t$), but also on the change of velocity with respect to position, $u\,\partial u/\partial x$.

4.45

4.45 Water flows steadily through the funnel show[redacted] Fig. P4.45. Throughout most of the funnel the flow is app[redacted] imately radial (along rays from O) with a velocity of $V =$ [redacted] where r is the radial coordinate and c is a constant. If the[redacted] locity is 0.4 m/s when $r = 0.1$ m, determine the acceler[redacted] at points A and B.



■ FIGURE P4.45

$V = \frac{c}{r^2}$

Assume that a good approximation of the flow field is along the radial coordinate towards the origin.

$\underline{v} = \frac{c}{r^2}\,\underline{i}_r$ and $v_r = \frac{c}{r^2}$

To find the acceleration, start with the material derivative of velocity

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}$$

The acceleration component in the radial direction is

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r}$$

Since $\partial v_r/\partial t = 0$ (no dependence of v_r on time t)

$$a_r = v_r \frac{\partial v_r}{\partial r}$$

$$\frac{\partial v_r}{\partial r} = \frac{-2c}{r^3}$$

Substituting into a_r

$$a_r = \left(\frac{c}{r^2}\right)\left(\frac{-2c}{r^3}\right) = \frac{-2c^2}{r^5}$$

To find c, use know the velocity $V_o = 0.4$ m/s when $r_o = 0.1$ m, and substitute into $v_r = c/r^2$

$$0.4 \text{ m/s} = \frac{c}{(0.1\text{ m})^2}$$

$$c = 0.004 \text{ m}^3/\text{s}$$

Thus

$$a_r = \frac{-2(0.004 \text{ m}^3/\text{s})^2}{r^5}$$

$$a_r = \frac{-(3.2\times10^{-5})\text{m}^6/\text{s}^2}{r^5}$$

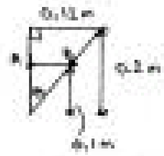
At point A:

$r = 0.1\ m$

$$a_r|_A = \frac{-3.2\times10^{-4}\ m^3/s}{(0.1\ m)^5}$$

$$\boxed{a_r|_A = -3.2\ m/s^2}$$

At point B:



$$r_B = \frac{0.1\ m}{\cos\theta}$$

To find θ:

$$\tan\theta = \frac{0.12\ m}{0.2\ m}$$

$$\theta = 30.964°$$

Therefore

$\cos\theta = 0.857$

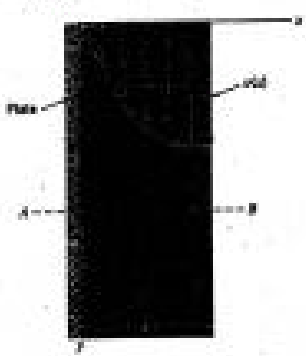
$r_B = 0.117\ m$

$$a_r|_B = \frac{-3.2\times10^{-4}\ m^3/s^2}{(0.117\ m)^5}$$

$$\boxed{a_r|_B = -1.48\ m/s^2}$$

4.55

4.55 A layer of oil flows down a vertical plate as shown in Fig. P4.55 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\,\hat{\mathbf{j}}$ where V_0 and h are constants. (a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer ($x = h$) is zero. (b) Determine the flowrate across surface AB. Assume the width of the plate is b. (Note: The velocity profile for laminar flow in a pipe has a similar shape. See Video V6.6.)



■ FIGURE P4.55

$$\underline{V} = \left(\frac{V_0}{h^2}\right)(2hx - x^2)\,\hat{j}$$

$$v = \left(\frac{V_0}{h^2}\right)(2hx - x^2)$$

(a) From the velocity profile, the velocity at $x = 0$ is zero:

$v(x=0) = 0$. Since the plate is located at $x = 0$ and is not moving, the fluid must stick to the plate. This is often referred to as the no slip boundary condition.

Shear stress:

$$\tau_{xy} = \mu\left(\frac{dv}{dx} + \cancel{\frac{du}{dy}}^{\,0}\right)$$

$$\frac{dv}{dx} = \left(\frac{V_0}{h^2}\right)(2h - 2x)$$

$$\frac{dv}{dx} = \frac{2V_0}{h^2}(h - x)$$

At the edge of the layer, $x = h$

thus

$$\left.\frac{dv}{dx}\right|_{x=h} = \frac{2V_0}{h^2}(h - h) = 0$$

Therefore, the shear stress at the edge of the layer is zero,

$$\tau = \mu(0) = 0$$

4.55 (b) Flowrate, Q, of a fluid is the velocity of the fluid multiplied by the cross sectional area of the area the fluid flows across. Here, the velocity is a function of x, so the flowrate is:

$$Q = \int_0^h b\, v(x)\, dx \qquad b = \text{width of the plate}$$

Substituting in $v(x)$:

$$Q = \int_0^h b\,(V_0/h^2)(2hx - x^2)\,dx$$

$$= b\,(V_0/h^2)\int_0^h (2hx - x^2)\,dx$$

$$= b\,(V_0/h^2)\left[2h\int_0^h x\,dx - \int_0^h x^2\,dx\right]$$

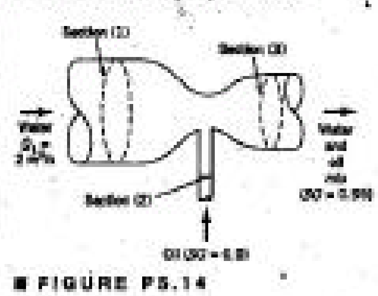
$$= b\,(V_0/h^2)\left[2h(\tfrac{1}{2})h^2 - \tfrac{1}{3}h^3\right]$$

$$= \frac{bV_0}{h^2}\left(h^3 - \tfrac{1}{3}h^3\right)$$

$$\boxed{Q = \tfrac{2}{3}\, b V_0 h}$$

5.14

5.14 Oil having a specific gravity of 0.9 is pumped as illustrated in Fig. P5.14 with a water jet pump (see Video V3.6). The water volume flowrate is 2 m³/s. The water and oil mixture has an average specific gravity of 0.95. Calculate the rate, in m³/s, at which the pump moves oil.



■ FIGURE P5.14

Find Q_2, the rate at which the pump moves oil.

Apply a mass balance:

$$\text{In} - \text{Out} + \underbrace{\cancel{\text{Generation}}}_{0} = \underbrace{\cancel{\text{Accumulation}}}_{0} \qquad \text{(No generation or accumulation)}$$

$$\text{In} = \text{Out}$$

$$\rho_{H_2O}Q_1 + (0.9)\rho_{H_2O}Q_2 = (0.95)\rho_{H_2O}\,Q_3 \qquad (1)$$

Divide by ρ_{H_2O}

$$Q_1 + (0.9)Q_2 = (0.95)\,Q_3 \qquad (2)$$

(2) is developed from the conservation of mass.

Water and oil can be considered incompressible, thus the volume of fluids that mix together is equal to the sum of the volumes of the individual fluids.

$$Q_1 + Q_2 = Q_3 \qquad (3)$$

Substituting in the expression for Q_3 from (3) into (2).

$$Q_1 + (0.9)Q_2 = 0.95\,(Q_1 + Q_2)$$

$$(1-0.95)Q_1 = (0.95-0.9)Q_2$$

$$0.05\,Q_1 = 0.05\,Q_2$$

Therefore:

$$\boxed{Q_2 = Q_1 = 2 \ m^3/s}$$

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Problem Set #3
Grading Guidelines
Michelle Wu

4.6 15 points total

- +2 Writing dy/dx = v/u
- +3 Finding $dy/dx = -2xy/(x^2 - y^2)$
- +5 Finding when flow is parallel to the y-axis, y = +/- x
- +5 Finding when flow is stationary

4.8 20 points total

- part (a) 10 points
- part (b) 10 points

4.23 15 points total

- part (a) 10 points
- part (b) 5 points

4.45 15 points total

4.55 20 points total

- part (a) 10 points
  - -2 Not shown that du/dx is equal to zero
  - -5 Not shown that at x=0, v=0 the fluid sticks to the plate
- part (b) 10 points
  - -3 Assumed the area was A = hb instead of setting up the integral. Velocity varies as a function of x, thus you need:

$$Q = \int_0^h bv(x)dx$$

  - -1 Incorrect limits for the integral Q

5.14 15 points total

Many people assumed that the ratio of the inflow of water and oil is 1:1, and thus the inflow rate of oil must equal the inflow rate of water. This assumption leads to the same answer, but generally, this is not the case and one point was deducted if no mathematical justifications were made in support of this.