10.301 Problem Set #4 Solutions

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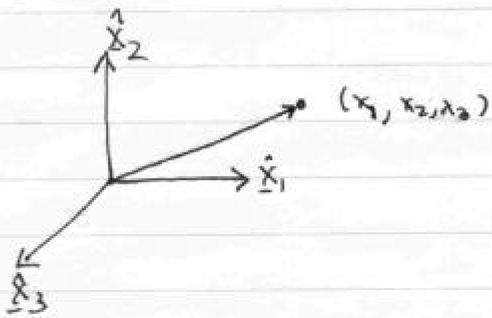
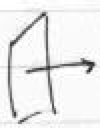
1)



The force on the test surface is given by

$$\underline{F} = A\,\underline{n} \cdot \underline{\underline{P}}$$

since we are evaluating $\underline{\underline{P}}$ at a point.

1.  $\underline{n} = \hat{\underline{x}}_1$, $\underline{F} = (1N)\,\hat{x}_1$

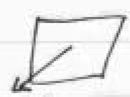
$$(1N)\,\hat{x}_1 = (1mm^2)\ \hat{x}_1 \cdot \underline{\underline{P}} = (1mm^2)\left[P_{11}\hat{\underline{x}}_1 + P_{12}\hat{\underline{x}}_2 + P_{13}\hat{\underline{x}}_3\right]$$

$$\therefore\ P_{11} = 1N/mm^2,\quad P_{12} = P_{13} = 0.$$

2.  $\underline{n} = \hat{x}_2$, $\underline{F} = (-2N)\,\hat{x}_3$

$$(-2N)\,\hat{x}_3 = (1mm^2)\,\hat{\underline{x}}_2 \cdot \underline{\underline{P}} = (1mm^2)\left[P_{21}\hat{x}_1 + P_{22}\hat{\underline{x}}_2 + P_{23}\hat{x}_3\right]$$

$$\therefore\ P_{23} = -2N/mm^2,\quad P_{21} = P_{22} = 0$$

3.  $\underline{n} = \hat{\underline{x}}_3$, $\underline{F} = (-2N)\,\hat{x}_2$

$$(-2N)\,\hat{\underline{x}}_2 = (1mm^2)\,\hat{\underline{x}}_3 \cdot \underline{\underline{P}} = (1mm^2)\left[P_{31}\hat{\underline{x}}_1 + P_{32}\hat{\underline{x}}_2 + P_{33}\hat{\underline{x}}_3\right]$$

$$\therefore\ P_{32} = -2N/mm^2,\quad P_{31} = P_{33} = 0$$

So the stress tensor has three nonzero components

$$\boxed{\underline{\underline{P}} = \frac{N}{mm^2}\left[\hat{\underline{x}}_1\hat{\underline{x}}_1 - 2\hat{\underline{x}}_2\hat{\underline{x}}_3 - 2\hat{\underline{x}}_3\hat{\underline{x}}_2\right]}$$

This is symmetric, as would be expected.

b) The normal vector to $\hat{\underline{x}}_1 + \hat{\underline{x}}_2$ must have unit length

$$\underline{n} = a(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)$$

$$|\underline{n}| = [a^2(1^2 + 1^2)]^{1/2} = 1 \quad \Rightarrow \quad a = 1/\sqrt{2}$$

Thus, the normal vector is $\underline{n} = \frac{1}{\sqrt{2}}(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)$

The force on the test surface is then

$$\underline{F} = A\,\underline{n}\cdot\underline{\underline{P}} = (1\,mm^2)\left[\frac{1}{\sqrt{2}}(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)\right]\cdot\left(\frac{N}{mm^2}\right)\left[\hat{\underline{x}}_1\hat{\underline{x}}_1 - 2\hat{\underline{x}}_2\hat{\underline{x}}_3 - 2\hat{\underline{x}}_3\hat{\underline{x}}_2\right]$$

$$\underline{F} = N\left[\frac{1}{\sqrt{2}}\hat{\underline{x}}_1 - \frac{2}{\sqrt{2}}\hat{\underline{x}}_3\right]$$

The net force is the magnitude of $\underline{F}$:

$$|\underline{F}| = \left\{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-2}{\sqrt{2}}\right)^2\right\}^{1/2} N = \left(\frac{1}{2} + 2\right)^{1/2} N = \sqrt{\frac{5}{2}}\, N$$

or $\boxed{F_{NET} = \sqrt{\frac{5}{2}}\, N}$

c) The force normal to the surface is

$$\underline{n} \cdot \underline{F} = \frac{1}{\sqrt{2}} (\hat{\underline{x}}_1 + \hat{\underline{x}}_2) \cdot \left[\frac{1}{\sqrt{2}} \hat{\underline{x}}_1 - \frac{2}{\sqrt{2}} \hat{\underline{x}}_3 \right] N$$

$$F_{normal} = -\frac{1}{2} N$$

d) To find the principal axes, we want a coordinate system $\underline{i}_1, \underline{i}_2, \underline{i}_3$ where

$$\underline{\underline{P}} = P_{11} \underline{i}_1 \underline{i}_1 + P_{22} \underline{i}_2 \underline{i}_2 + P_{33} \underline{i}_3 \underline{i}_3$$

From the solution to part a, we already know $\hat{\underline{x}}_1$ is a principal direction, i.e. $\underline{i}_1 = \hat{\underline{x}}_1$. The cross terms $\hat{\underline{x}}_2 \hat{\underline{x}}_3$ represent shear in the plane. Example 1 in the Stress Notes shows how the principal axes can be found. Rotating the $\hat{\underline{x}}_2, \hat{\underline{x}}_3$ axes by 45° around $\hat{\underline{x}}_1$, from (3.4)

$$\hat{\underline{x}}_2 = \frac{\sqrt{2}}{2} (\underline{i}_2 - \underline{i}_3) \; ; \quad \hat{\underline{x}}_3 = \frac{\sqrt{2}}{2} (\underline{i}_2 + \underline{i}_3)$$

which can be solved for

$$\underline{i}_2 = \frac{\sqrt{2}}{2} (\hat{\underline{x}}_2 + \hat{\underline{x}}_3) \; ; \quad \underline{i}_3 = \frac{\sqrt{2}}{2} (\hat{\underline{x}}_3 - \hat{\underline{x}}_2)$$

and we have

$$\hat{\underline{x}}_2 \hat{x}_3 + \hat{\underline{x}}_3 \hat{x}_2 = \underline{i}_2 \underline{i}_2 - \underline{i}_3 \underline{i}_3$$

So that the principal stresses are

$$\underline{\underline{P}} = \frac{N}{mm^2} \left[\underline{i}_1 \underline{i}_1 - 2 \underline{i}_2 \underline{i}_2 + 2 \underline{i}_3 \underline{i}_3 \right]$$

where $\underline{i}_1 = \hat{\underline{x}}_1$, $\underline{i}_2 = \frac{\sqrt{2}}{2} (\hat{x}_2 + \hat{x}_3)$, $\underline{i}_3 = \frac{\sqrt{2}}{2} (\hat{x}_2 - \hat{x}_3)$.

Note that the algebraic sign of the unit vectors in the principal axis system is not important, since the principal stresses involve the <u>product</u> of these directions That is why we could change the sign of $\underline{i}_3$ during this problem.

The principal stresses are tensile in the $\underline{i}_1\underline{i}_1$ and $\underline{i}_3\underline{i}_3$ directions and compressive in the $\underline{i}_2\underline{i}_2$ direction.

2.) In a coordinate system with unit vectors $\underline{i}_1, \underline{i}_2$, and $\underline{i}_3$ the stress tensor is

$$\underline{\underline{P}} = 2\underline{i}_1\underline{i}_1 - \underline{i}_1\underline{i}_2 + \underline{i}_1\underline{i}_3 - \underline{i}_2\underline{i}_1 + \underline{i}_2\underline{i}_3 + \underline{i}_3\underline{i}_1 + \underline{i}_3\underline{i}_2 + 2\underline{i}_3\underline{i}_3$$

We want the principle stresses so that $\underline{\underline{P}}$ can be written as a sum of normal stresses:

$$\underline{\underline{P}} = \lambda_1 \underline{n}^1\underline{n}^1 + \lambda_2 \underline{n}^2\underline{n}^2 + \lambda_3 \underline{n}^3\underline{n}^3 \,; \quad \underline{n}^j = n_1^j \underline{i}_1 + n_2^j \underline{i}_2 + n_3^j \underline{i}_3$$

The various λ_j and $\underline{n}^j$ are found from the solution of the problem

$$\underline{\underline{P}} \cdot \underline{n}^j = \lambda_j \underline{n}^j$$

Using the vector identity

$$\underline{\underline{I}} \cdot \underline{n} = \underline{n}$$

We can write

$$\underline{\underline{P}} \cdot \underline{n}^j = \underline{\underline{I}} \lambda_j \cdot \underline{n}^j$$

$$(\underline{\underline{P}} - \underline{\underline{I}}\lambda_j) \cdot \underline{n}^j = \underline{0}$$

In components:

$$\left[\begin{array}{l}(2-\lambda_j)\underline{i}_1\underline{i}_1 - \underline{i}_1\underline{i}_2 + \underline{i}_1\underline{i}_3 - \underline{i}_2\underline{i}_1 - \lambda_j\underline{i}_2\underline{i}_2 \\ + \underline{i}_2\underline{i}_3 + \underline{i}_3\underline{i}_1 + \underline{i}_3\underline{i}_2 + (2-\lambda_j)\underline{i}_3\underline{i}_3\end{array}\right] \cdot \left[n_1^j \underline{i}_1 + n_2^j \underline{i}_2 + n_3^j \underline{i}_3\right] = \underline{0}$$

Performing the dot product

$$[(2-\lambda_j)n_1^j - n_2^j + n_3^j]\,\underline{i}_1 + [-n_1^j - \lambda_j n_2^j + n_3^j]\,\underline{i}_2$$
$$+ [n_1^j + n_2^j + (2-\lambda_j)]\,\underline{i}_3 = 0\,\underline{i}_1 + 0\,\underline{i}_2 + 0\,\underline{i}_3$$

Equating the components of this vector eqn

$$\begin{aligned} (2-\lambda_j)n_1^j - n_2^j + n_3^j &= 0 \\ -n_1^j - \lambda_j n_2^j + n_3^j &= 0 \\ n_1^j + n_2^j + (2-\lambda_j)n_3^j &= 0 \end{aligned}$$

This eqn set has nontrivial solutions if

$$\det \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Forming the determinant

$$(2-\lambda)(\lambda+1)(\lambda+3) = 0$$

so we have $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$. To compute the directions of these stresses, we need to substitute the λ and solve for the n. These are not unique, so we also require (on a physical basis) that $\underline{n}$ be normalized,

$$|\underline{n}| = (n_1^2 + n_2^2 + n_3^2)^{1/2} = 1.$$

The normalization of $\underline{n}$ insures that the λ correspond to the magnitude of the stresses.

$\lambda = -1$:
$$\left.\begin{array}{l} 3n_1^1 - n_2^1 + n_3^1 = 0 \\ -n_1^1 + n_2^1 + n_3^1 = 0 \\ n_1^1 + n_2^1 + 3n_3^1 = 0 \\ (n_1^1)^2 + (n_2^1)^2 + (n_3^1)^2 = 1 \end{array}\right\} \quad n_1^1 = \frac{-1}{\sqrt{6}},\ n_2^1 = \frac{-2}{\sqrt{6}},\ n_3^1 = \frac{1}{\sqrt{6}}$$

$\lambda = 2$:
$$\left.\begin{array}{l} -n_2^2 + n_3^2 = 0 \\ -n_1^2 - 2n_2^2 + n_3^2 = 0 \\ n_1^2 + n_2^2 = 0 \\ (n_1^2)^2 + (n_2^2)^2 + (n_3^2)^2 = 1 \end{array}\right\} \quad n_1^2 = \frac{-1}{\sqrt{3}},\ n_2^2 = \frac{1}{\sqrt{3}},\ n_3^2 = \frac{1}{\sqrt{3}}$$

$\lambda = 3$
$$\left.\begin{array}{l} -n_1^3 - n_2^3 + n_3^3 = 0 \\ -n_1^3 - 3n_2^3 + n_3^3 = 0 \\ n_1^3 + n_2^3 - n_3^3 = 0 \\ (n_1^3)^2 + (n_2^3)^2 + (n_3^3)^2 = 1 \end{array}\right\} \quad n_1^3 = \frac{1}{\sqrt{2}},\ n_2^3 = 0,\ n_3^3 = \frac{1}{\sqrt{2}}$$

Consequently, the stress tensor in principal axes is

$$\boxed{\underline{\underline{P}} = -\underline{n}^1\underline{n}^1 + 2\underline{n}^2\underline{n}^2 + 3\underline{n}^3\underline{n}^3}$$

where the directions are

$$\boxed{\begin{array}{l} \underline{n}^1 = \frac{-1}{\sqrt{6}}\,\underline{i}_1 - \frac{2}{\sqrt{6}}\,\underline{i}_2 + \frac{1}{\sqrt{6}}\,\underline{i}_3 \\ \underline{n}^2 = \frac{-1}{\sqrt{3}}\,\underline{i}_1 + \frac{1}{\sqrt{3}}\,\underline{i}_2 + \frac{1}{\sqrt{3}}\,\underline{i}_3 \\ \underline{n}^3 = \frac{1}{\sqrt{2}}\,\underline{i}_1 + \frac{1}{\sqrt{2}}\,\underline{i}_3 \end{array}}$$

As was the case in problem 1, we could have also used $-\underline{n}^1$, $-\underline{n}^2$, or $-\underline{n}^3$ to define the directions of these axes since only their products appear in $\underline{\underline{T}}$.

The principal stress is compressive in $\underline{n}^1\underline{n}^1$ and tensile in $-\underline{n}^2\underline{n}^2$ and $\underline{n}^3\underline{n}^3$.

b) The eigenvalues of the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

are computed by the solution of

$$\det \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 0$$

This is just what we had with the previous solution using the invariant notation, so the eigen values ~~and~~ are

$$\lambda_1 = -1, \ \lambda_2 = 2, \ \lambda_3 = 3.$$

The eigen vectors corresponding to these λ are the nontrivial solutions to

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \lambda \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Upon requiring that the eigenvectors be normalized to unity, we get the previous result:

$$\lambda_1 \rightarrow \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} ; \quad \lambda_2 \rightarrow \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} ; \quad \lambda_3 \rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

The reason that these eigenvalues & vectors are the same is that the vector / dyadic representation, while more physical, can be equivalently represented as matrices. In the previous section, the term $\underline{\underline{P}} \cdot \underline{n}$ is like a force, whose value is λ and whose direction is $\underline{n}$.

In summary, the physical meaning of the principal stresses are the coordinates in which there are no mixed terms (i.e. all normal stresses). The eigenvalue / eigenvector calculation is a formal mathematical procedure for determining these directions and stresses.