10.301- Review Problems for Exam III
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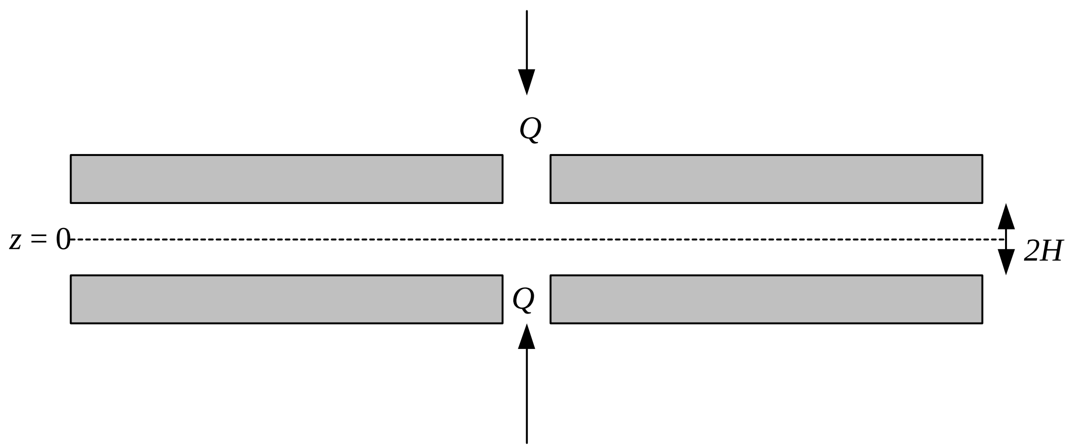
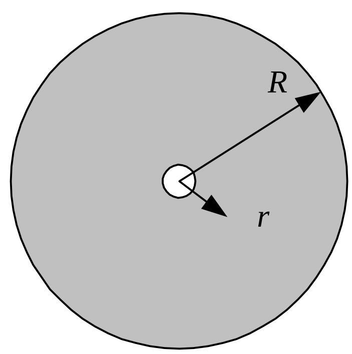
Problem 1. Flow between two radial disks (from W.M. Deen)

Consider the flow between two radial disks of radius *R* separated by a distance *2H* with a concentric hole. Fluid flows through the hole on both sides at a volumetric flow rate *Q*, moving out towards the edges of the disk. Assume that the flow is steady and the fluid is Newtonian.





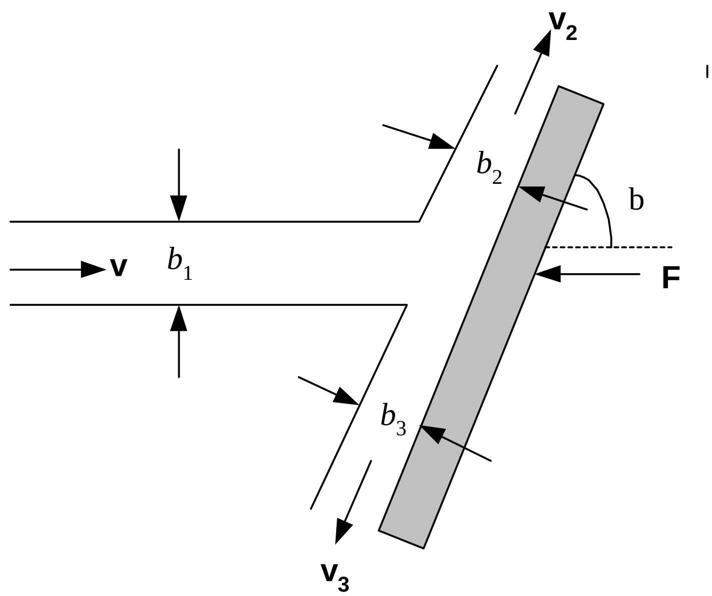
(a) Relate the flow rate *Q* to the average fluid velocity *U(r)*.

(b) Assume a velocity of the form $\mathbf{v} = v_r(r,z)\mathbf{i_r} = f(z)/r\ \mathbf{i_r}$ and a dynamic pressure field $P=P(r)$. Show that this satisfies continuity and the Navier-Stokes equation and compute the velocity field.

(c) Eliminate the pressure from the expression in part (b), thereby expressing v_r in known quantities. What is the function $f(z)$?

Problem 2. Fluid Jet Impacting an Inclined Plate (K. Smith)

Consider a fluid jet of velocity **v** impacting upon a plate inclined at an angle β with respect to the incoming jet. A force **F** is applied normal to the plate to hold it in place. Assuming that viscous effects, gravity, and pressure are negligible compared to inertia, compute the widths (b_1, b_2, and b_3) and velocities ($\mathbf{v}_2$ and $\mathbf{v}_3$) of each jet. You may assume to process is uniform in the *z*-direction, so that the problem is essentially two-dimensional. Also, assume that the velocities are uniform across their cross-sections.



Problem 1

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a) Relate $U(r)$ to Q.

$$\underset{\substack{\uparrow \\ \text{total} \\ \text{flowrate}}}{2Q} = \underset{\substack{\uparrow \\ \text{cross-sectional} \\ \text{area}}}{\left[2\pi r\,(2H)\right]}\ \underset{\substack{\uparrow \\ \text{mean velocity}}}{U(r)}$$

$$\boxed{U(r) = \frac{Q}{2\pi H r}}$$

b) $\underline{V} = V_r(r,z)\, i_r = f(z)/r\ 1_r$; $P = P(r)$

i) continuity: $\dfrac{\partial}{\partial r}(r v_r) = 0 \Rightarrow V_r = \dfrac{f(z)}{r}$ ✓

ii) θ-momentum: $\dfrac{\partial P}{\partial \theta} = 0 \Rightarrow P = P(r,z)$

(iii) z-momentum: $\dfrac{\partial P}{\partial z} = 0 \Rightarrow P = P(r)$ ✓

(iv) r-momentum:

$$0 = -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\underset{=f(z)}{(r v_r)}\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\mu}\frac{dP}{dr}$$

↑ not a function of z.

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Integrate $\frac{\partial v_r}{\partial z} = \frac{z}{\mu}\frac{dP}{dr} + c_1(r)$

$\uparrow = 0$ by symmetry @ centerline

Integrate: $v_r = \frac{z^2}{2\mu}\frac{dP}{dr} + c_2(r)$

No slip at $z = H$.

$$0 = \frac{H^2}{2\mu}\frac{dP}{dr} + c_2(r) \Rightarrow c_2(r) = \frac{-H^2}{2\mu}\frac{dP}{dr}$$

Consequently,

we still do not know this!

$$\boxed{v_r = \frac{-H^2}{2\mu}\frac{dP}{dr}\left[1 - \left(\frac{z}{H}\right)^2\right]}$$

c) Find the function $f(z)$ by relating to $U(r)$

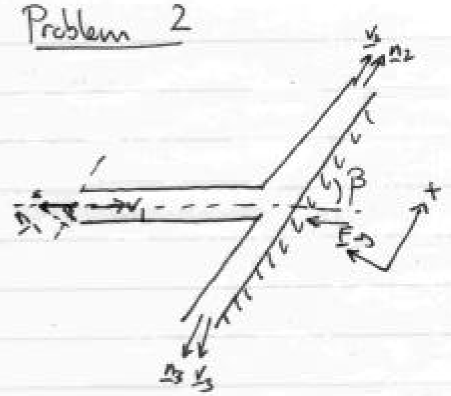
$$U(r) = \frac{1}{2H}\int_{-H}^{H} v_r\, dz = \frac{1}{2H}\left[\frac{-H^2}{2\mu}\frac{dP}{dr}\right]\int_{-H}^{H} 1 - \frac{z^2}{H^2}\, dz$$

$$U(r) = \frac{-H^2}{2\mu}\frac{dP}{dr}\left(\frac{2}{3}\right) = \frac{Q}{2\pi H r}$$

So $\frac{-H^2}{2\mu}\frac{dP}{dr} = \frac{3Q}{4\pi H r}$

and $\boxed{v_r = \frac{1}{r}\left[\underbrace{\frac{3Q}{4\pi H}\left(1 - \frac{z^2}{H^2}\right)}_{f(z)}\right]}$

Problem 2 K. Dorfman



1. $-\underline{n}\cdot\underline{v}_1 = \cancel{n}\ v_1 = v$
2. $-\underline{n}\cdot\underline{v}_2 = -v_2$
3. $-\underline{n}\cdot\underline{v}_3 = -v_3$

Conservation of mass: $0 = \int_{\partial\mathcal{R}} (-\underline{n}\cdot\underline{v})\, dS$

$$0 = v b_1 \cancel{+} - v_2 b_2 - v_3 b_3$$

Since we are neglecting viscous effects, pressure, and gravity, (The pressure are atmospheric at the edges) then Bernoulli's equation says that $\frac{1}{2}\rho v^2$ is a constant on a streamline. Thus:

$\cancel{v}$ $v_1 = v_2 = v_3$ (magnitudes are equal)

Substitute this into the mass balance and we get

$$b_1 = b_2 + b_3$$

To use conservation of momentum, we need the vector forms of the velocities and the force:

$$\boxed{\begin{aligned} \underline{v}_1 &= v(\cos\beta\, \underline{i}_x - \sin\beta\, \underline{i}_y) \\ \underline{v}_2 &= v\, \underline{i}_x \\ \underline{v}_3 &= -v\, \underline{i}_x \end{aligned}}$$

$$\underline{F} = F \underline{i}_y$$

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Neglecting the indicated forces, the linear momentum equation adopts the form:

$$\underline{0} = \int_{\partial\Omega} (-\underline{n}\cdot\underline{v})\,\rho\underline{v}\, dS + \underline{F}$$

$$= \int_1 v\,\rho\underline{v}_1\, dS + \int_2 (-v)\,\rho\,\underline{v}_2\, dS + \int_3 (-v)\rho\underline{v}_3\, dS + F\underline{i}_y$$

$$0\underline{i}_x + 0\underline{i}_y = \rho v^2 b_1(\cos\beta\,\underline{i}_x - \sin\beta\,\underline{i}_y) + (-v)\,\rho v b_2\underline{i}_x + (-v)\rho(-v\underline{i}_x)b_3 + F\underline{i}_y$$

For the y-component:

$$0 = -\rho v^2 b_1 \sin\beta + F_y \Rightarrow \boxed{b_1 = \frac{F_y}{\rho v^2 \sin\beta}}$$

For the x-component:

$$0 = \rho v^2 \cos\beta\, b_1 - \rho v^2 b_2 + \rho v^2 b_3 = 0$$

$$0 = b_1\cos\beta - b_2 + b_3$$

$$0 = b_1 \cos\beta - b_2 + b_1 - b_2 \qquad \text{(from } b_3 = b_1 - b_2 \text{ from c. mass)}$$

$$\boxed{b_2 = \frac{b_1(1+\cos\beta)}{2}}$$

$$\boxed{b_3 = b_1 - b_2 = \frac{b_1(1-\cos\beta)}{2}}$$