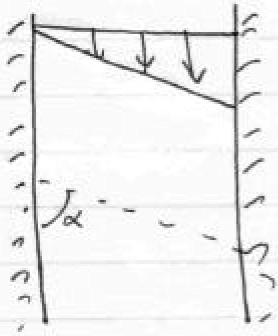
Solutions to Practice Problems for Exam 2 — Kevin Dorfman

1).



$\underline{V} = V_0 \frac{x}{l} \underline{i}_y$

a) $\nabla \cdot \underline{V} \overset{?}{=} 0$

$$\left(\underline{i}_x \frac{\partial}{\partial x} + \underline{i}_y \frac{\partial}{\partial y} + \underline{i}_z \frac{\partial}{\partial z} \right) \cdot \frac{V_0 x}{l} \underline{i}_y \overset{?}{=} 0$$

$$\frac{\partial}{\partial y} \frac{V_0 x}{l} \overset{?}{=} 0$$

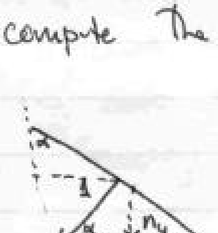
$$0 = 0$$

$\boxed{\text{The flow is incompressible.}}$

b) The flow rate through the surface is given by

$$dQ = \underline{n} \cdot \underline{v} \, dA$$

To compute the normal $\underline{n} = n_x \underline{i}_x + n_y \underline{i}_y$

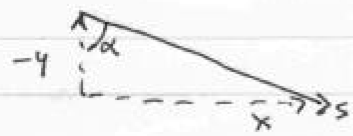


$n_x = -\cos\alpha$

$n_y = \sin\alpha$

$$\underline{n} \cdot \underline{v} = \left(-\underline{i}_x \cos\alpha + \underline{i}_y \sin\alpha \right) \cdot \left(\frac{V_0 x}{l} \underline{i}_y \right) = \frac{V_0 x}{l} \sin\alpha$$

It will be easier to compute dA by using the line of the surface (s)



$X = s \cdot \sin\alpha$

$$\int dQ = \int \underline{n} \cdot \underline{v} \, dA$$

$$Q = \int_{x=0}^{x=\ell} \frac{V_0 X}{\ell} \sin\alpha \, ds$$

$$= \int_{s=0}^{s=\ell/\sin\alpha} \frac{V_0}{\ell} \sin^2\alpha \; s \, ds$$

$$= \frac{V_0}{\ell} \sin^2\alpha \left[\frac{s^2}{2} \right]_0^{\ell/\sin\alpha}$$

$$\boxed{Q = \frac{b \ell V_0}{2}}$$

c) From part a, we already know $\nabla \cdot \underline{v} = 0$. To compute $\nabla \underline{v}$:

$$\nabla \underline{v} = \left(\underline{i}_x \frac{\partial}{\partial x} + \underline{i}_y \frac{\partial}{\partial y} + \underline{i}_z \frac{\partial}{\partial z} \right) \left(\frac{V_0 x}{\ell} \underline{i}_y \right)$$

$$\nabla \underline{v} = \underline{i}_x \underline{i}_y \frac{V_0}{\ell}$$

The transpose is $(\nabla \underline{v})^t = \underline{i}_y \underline{i}_x \frac{V_0}{\ell}$

Consequently, the deviatoric stress is

$$\underline{\underline{\tau}} = \frac{\eta v_0}{\ell}\left(\underline{i}_x \underline{i}_y + \underline{i}_y \underline{i}_x\right)$$

d) The force exerted by the stress across the surface is

$$d\underline{F} = \underline{n} \cdot \underline{\underline{P}}\, dA$$

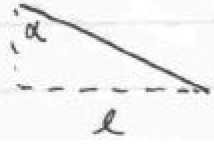
$$d\underline{F} = \left(-\underline{i}_x \cos\alpha + \underline{i}_y \sin\alpha\right) \cdot \left(\frac{\eta v_0}{\ell}\, \underline{i}_x \underline{i}_y + \underline{i}_y \underline{i}_x\right) dA$$

$$\int d\underline{F} = \int \frac{\eta v_0}{\ell}\left(\underline{i}_x \sin\alpha - \underline{i}_y \cos\alpha\right) dA$$

$$\underline{F} = \frac{\eta v_0}{\ell}\left(\underline{i}_x \sin\alpha - \underline{i}_y \cos\alpha\right) A$$

If the surface has a depth, d, into the page, then

$$A = \frac{d\,\ell}{\sin\alpha} \Rightarrow \boxed{\underline{F} = \eta v_0 d\left(\underline{i}_x - \underline{i}_y \cot\alpha\right)}$$



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2) The acceleration is computed by

$$\underline{a} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

$$\underline{a} = \frac{\partial}{\partial t}\left(u\underline{i}_x + v\underline{i}_y\right) + \underbrace{(u\underline{i}_x + v\underline{i}_y)\cdot\left(\underline{i}_x\frac{\partial}{\partial x} + \underline{i}_y\frac{\partial}{\partial y}\right)}_{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}}(u\underline{i}_x + v\underline{i}_y)$$

$$\underline{a} = \underline{i}_x\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] + \underline{i}_y\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]$$

Computing these derivatives:

$$\frac{\partial u}{\partial t} = A\omega^2 e^{ky} \cancel{\cos} \sin(kx - \omega t) = \omega v$$

$$\frac{\partial u}{\partial x} = A\omega e^{ky}(-k) \cancel{\cos} \sin(kx - \omega t) = -kv$$

$$\frac{\partial u}{\partial y} = A\omega (k) e^{ky} \cos(kx - \omega t) = ku$$

$$\frac{\partial v}{\partial t} = -A\omega^2 e^{ky} \cos(kx - \omega t) = -\omega u$$

$$\frac{\partial v}{\partial x} = +A\omega e^{ky} k \cos(kx - \omega t) = +ku$$

$$\frac{\partial v}{\partial y} = A\omega k e^{ky} \sin(kx - \omega t) = kv$$

$$\underline{a} = \underline{i}_x\left[\omega v + u(-kv) + v(ku)\right] + \underline{i}_y\left[-\omega u + u(+ku) + v(kv)\right]$$

$$\boxed{\underline{a} = \underline{i}_x\,\omega v + \underline{i}_y\left[u(ku - \omega) + kv^2\right]}$$

b) The streamlines are the solutions to

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dy}{dx} = \frac{v}{u} = \frac{\sin(kx - \omega t)}{\cos(kx - \omega t)}$$

If we are interested in time $t=0$

$$\frac{dy}{dx} = \frac{\sin kx}{\cos kx}$$

$$dy = -\frac{1}{k} \frac{d(\cos kx)}{\cos(kx)}$$

$$\boxed{y = -\frac{1}{k} \ln[\cos(kx)] + c}$$

The constant c is found from the point we are looking at.

$(0,0) \Rightarrow 0 = -\frac{1}{k} \ln(1) + c \Rightarrow c = 0$

$(0, -2\pi/k) \Rightarrow -\frac{2\pi}{k} = -\frac{1}{k} \ln 1 + c \Rightarrow c = -\frac{2\pi}{k}$