13.012 Hydrodynamics for Ocean Engineers

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Lecture #4: Archimedes's Principle and Static Stability
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Handouts:
1. Supplemental notes
2. HW #3

Reading: Chapter 2; Notes.

I. Archimedes's Principle:

The force on a body due to pressure alone (in the absence of viscous forces)

$$\vec{F} = \iint_S p\,\hat{n}\,ds$$

where pressure is a function of depth below the free surface:

$$p(z) = -\rho g z\,.$$

$$\vec{F} = \iint_S p\;\hat{n}\,ds = -\rho g \iint_S z\;\hat{n}\,ds$$

By Calculus the surface integral can be converted into a volume integral:

$$\iint_S z\,\hat{n}\,ds = \iiint_V \nabla z\;dV$$

So that the force becomes:

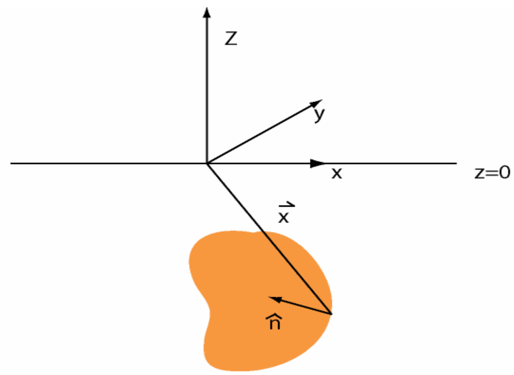
$$\vec{F} = \underline{\iint_S p\,\hat{n}\,ds} = -\rho g \iint_S z\,\hat{n}\,ds = \rho g \iiint_v dV = \underline{\rho g V\,\hat{k}}$$

We can see now that the buoyancy force acts to counterbalance the displaced volume of fluid. For a half submerged body the area of the water plane must be accounted for in the integration.

II. Moment on a body (Ideal Fluid)

The moment on a submerged body follows directly from structural mechanics or dynamics methodologies.

$$\vec{M} = \iint_S p(\vec{x} \times \hat{n})\, ds \equiv (M_1, M_2, M_3) = (M_x, M_y, M_z)$$



Recall from vector calculus:

$$\vec{x} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{array}{r} \hat{i}(y n_3 - z n_2) \\ -\hat{j}(x n_3 - z n_1) \\ +\hat{k}(x n_2 - y n_1) \end{array}$$

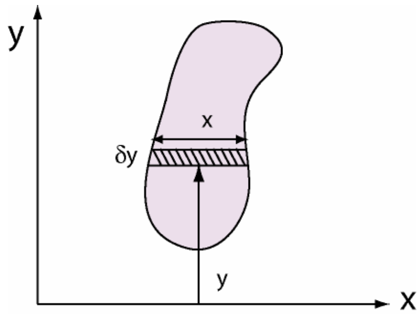
So we get the moments as follows:

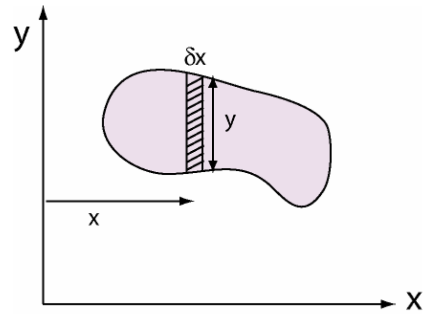
$$M_1 = \iint_s p(y n_3 - z n_2)\, ds \text{ about the x-axis}$$

We can also calculate in a similar fashion: M_2 about the y-axis and M_3 about the z-axis.

III. Center of Buoyancy

Calculating the Center of buoyancy, it is first necessary to find the center of area.





Area: $A = \int y\, dx$

When calculating a ships area, an approximate area is used since there is often no mathematical formula for the ships geometry.

First moment of the area is found as follows:

$$M_{yy} = \int y_1\, x\, dx$$

$$M_{xx} = \int x_1\, y\, dy$$

The coordinated for the center of area are then found using the above equations:

$$\bar{x} = \frac{M_{yy}}{A} = \frac{1}{A}\int x\, y_1\, dx$$
$$\bar{y} = \frac{M_{xx}}{A} = \frac{1}{A}\int y\, x_1\, dy$$

Center of buoyancy is the point at which the buoyancy force acts on the body and is equivalently the geometric center of the submerged portion of the hull.

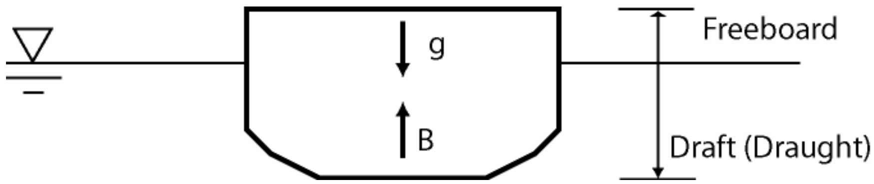
IV. Static Equilibrium: Gravity and buoyancy balance

A body at rest is considered to be in equilibrium.

The mass of the object (vessel) is equal to the mass of an equivalent volume of water (displacement). When a vessel is in static equilibrium, the center of gravity and buoyancy act on the same vertical line. Buoyancy acts perpendicular to the water line (upwards) at the center of buoyancy.

Center of gravity usually lies amidships and near to the water line.

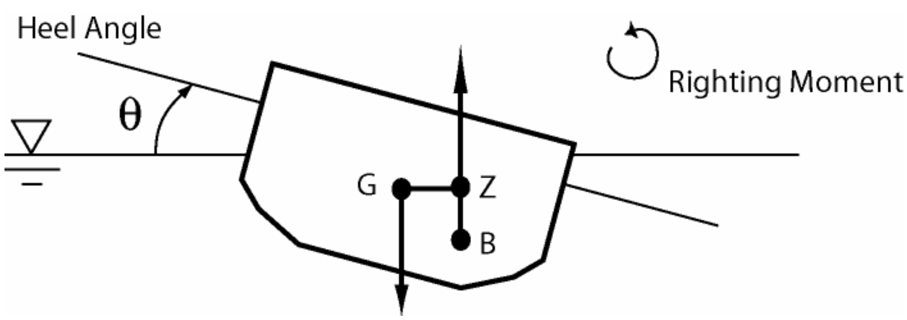
Center of buoyancy changes dependant on the volume of the submerged portion of the vessel. As a ship rolls, pitches, etc. the center of buoyancy moves according to the "new" shape of the submerged block. Take for example a container ship: as containers are loaded the weight of the ship increases and it submerges further into the water. With the increased draft the center of buoyancy shifts.



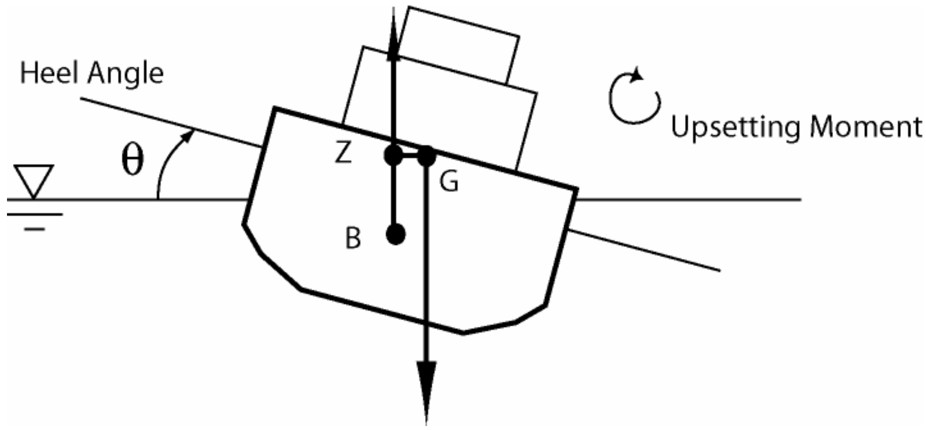
A vessel in static equilibrium with the centers of gravity and buoyancy aligned in the same vertical line.

V. Righting moments: The moment required/available to right a vessel.

An object disturbed from a static position that returns to its original position is considered statically stable or in *static equilibrium.* Such vessels are "self-righting". However if the vessel is unable to return to its equilibrium position and continues to turn over (capsize) it is considered unstable. It a vessel is in equilibrium at any position then it is neutrally stable (take for example a simple cylinder on its side – turn it to any angle and it will stay).



A statically stable vessel with a positive righting arm.



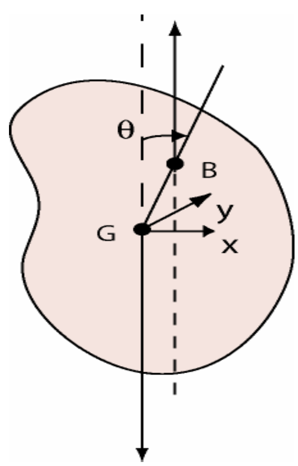
Statically unstable vessel with a negative righting arm.

The righting moment is equivalent to the Displacement of the vessel times the righting arm:

$$RM = W \times \overline{GZ}$$

VI. Small rotation about the y-axis equation for stability:

Sum of the torques about the center of rotation must be zero (conservation of angular momentum!)



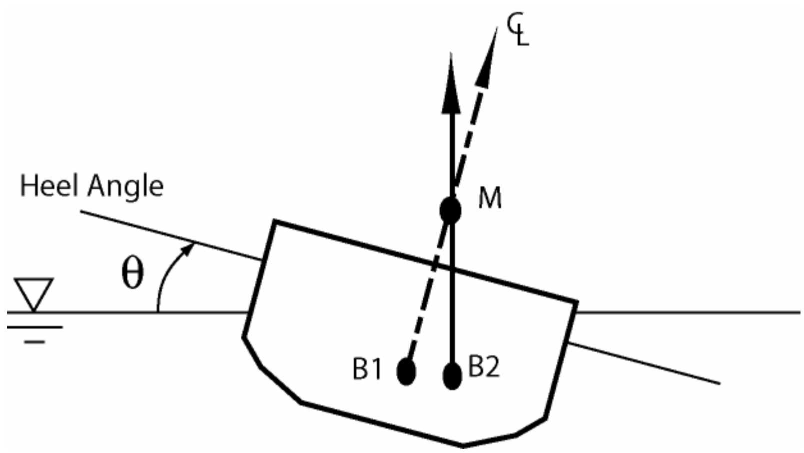
We can write the equation of motion in terms of the rotational displacement, θ, and rotational acceleration, $\ddot{\theta}$:

$$I\ddot{\theta} + l\ddot{\theta} = 0$$

where $l = |GB|$. This is the equation of simple harmonic motion. The motion is in stable equilibrium when $l > 0$ and unstable when $l < 0$.

VII. Metacenter: A ships metacenter is the intersection of two lines of action of the buoyancy force.

What does this mean? Well, basically it means that as a ship rolls (heels) through an angle, θ, the center of buoyancy, *B,* shifts along a semi-elliptical path (depending on geometry). Since the buoyancy always acts vertically, the two lines of action for two different heel angles must cross at a point. This point is the metacenter of the ship, denoted by *M* in the figure below.



The metacenter is the intersection of two distinct lines of action of the buoyancy force.

When on at an angle of roll (heel) the center of buoyancy and center of gravity no longer act on the same vertical line of action.

Transverse Metacenter: Roll/Heel
Longitudinal Metacenter: Pitch/Trim

The concept of the metacenter is only really valid for small angles of motion.

Metacentric height is the distance measured from the metacenter to the center of gravity, $\overline{GM}$. If the metacentric height is large, then the vessel is considered to be "stiff" in roll – indicating that there will be a large righting moment as a result of small roll angles. In contrast, if the metacentric height is small then the ship rolls slowly due to a smaller righting arm.

