13.012 READING 7:
FLUID FORCES ON BODIES

FALL 2003

1. Steady Flow

In order to design offshore structures, surface vessels and underwater vehicles, an understanding of the basic fluid forces acting on a body is is needed. In the case of steady viscous flow, these forces are straightforward. *Lift* force, perpendicular to the velocity, and *Drag* force, inline with the flow, can be calculated based on the fluid velocity, U, force coefficients, C_D and C_L, object dimensions or area, A, and fluid density, ρ.

$$F_{Drag} = \frac{1}{2}\rho U^2 A\ C_D \tag{1.1}$$

$$F_{Lift} = \frac{1}{2}\rho U^2 A\ C_L \tag{1.2}$$

These equations can also be used in a quiescent (stationary) fluid for a steady translating body, where U is the body velocity instead of the fluid velocity.

At this point it might be good to review some concepts covered in 13.021. Chapter 4 in Newman's *Marine Hydrodynamics* offers a solid presentation of this material as well.

2. Unsteady Motion

Beyond steady flow, we must consider unsteady, time dependent motions of both the fluid and the body. Fluid inertial forces arise that add to the total forcing on a body.

Take the case of an unsteady moving body, $U_b(t)$, in an unbounded, inviscid fluid ($\mu = 0$) with zero velocity, $U_f = 0$. The time-dependent force on the body is directly proportional to the body acceleration:

$$F(t) = -m_a \frac{dU_b(t)}{dt} \tag{2.1}$$

where m_a depends on the body geometry and direction of motion. This is an *added* inertial force, or *added mass* force on the body. By comparison, in an inviscid steady flow, by D'Alembert's Paradox, the force on the body would be zero.

Looking at a body in two-dimensions we can have linear motion in two directions and rotational motion in one direction. (Think of these coordinates as if you were looking down on a ship.)

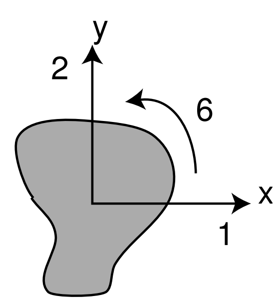


FIGURE 1. Two dimensional motion with axis (x,y) fixed on the body. 1: Surge, 2: Sway, 6: Yaw

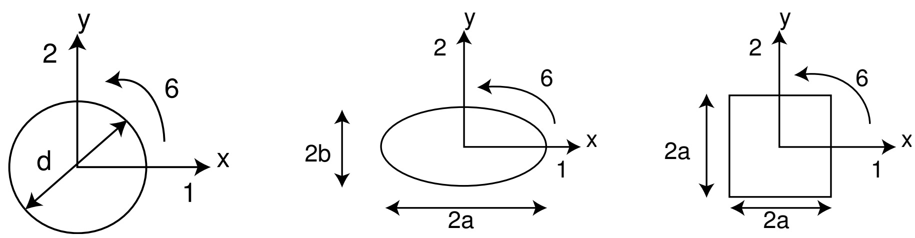
The unsteady forces on the body in the three directions are:

$$\begin{aligned}
-F_1 &= m_{11}\frac{du_1}{dt} + m_{12}\frac{du_2}{dt} + m_{16}\frac{du_6}{dt} \\
-F_2 &= m_{21}\frac{du_1}{dt} + m_{22}\frac{du_2}{dt} + m_{26}\frac{du_6}{dt} \\
-F_6 &= m_{61}\frac{du_1}{dt} + m_{62}\frac{du_2}{dt} + m_{66}\frac{du_6}{dt}
\end{aligned} \tag{2.2}$$

This set of equations can be written in matrix form, $\underline{F} = [M]\underline{\dot{u}}$,

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ m_{61} & m_{62} & m_{66} \end{bmatrix} \begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \\ \frac{du_6}{dt} \end{pmatrix} \tag{2.3}$$

For symmetric geometries the added mass tensor simplifies significantly. For example, figure 2 shows added mass values for a circle, ellipse, and square. In the case of the circle and square, movement in the 1 and 2 directions yields similar geometry and identical added mass coefficients ($m_{11} = m_{22}$).



Circle

$m_{11} = m_{22} = \rho\frac{\pi}{4}d^2$
$m_{66} = 0$

Ellipse

$m_{11} = \rho\pi b^2$
$m_{22} = \rho\pi a^2$
$m_{66} = \rho\frac{\pi}{8}\left(a^2 - b^2\right)^2$

Square

$m_{11} = m_{22} = 1.51\pi a^2$
$m_{66} = 0.234\rho\pi a^4$

FIGURE 2. Two dimensional added mass coefficients for a circle, ellipse, and square in 1: Surge, 2: Sway, 6: Yaw

Using these coefficients and those tabulated in Newman's *Marine Hydrodynamics* on page 145 we can determine the added mass forces quite simply.

Kinetic Energy: The kinetic energy in the flow is found as

$$KE = \frac{1}{2}m_a U^2 \tag{2.4}$$

Momentum: The fluid momentum is found as

$$Momentum = m_a U \tag{2.5}$$

Added Mass Force: The added mass force is

$$F_{ma} = -m_a \frac{dU}{dt} \tag{2.6}$$

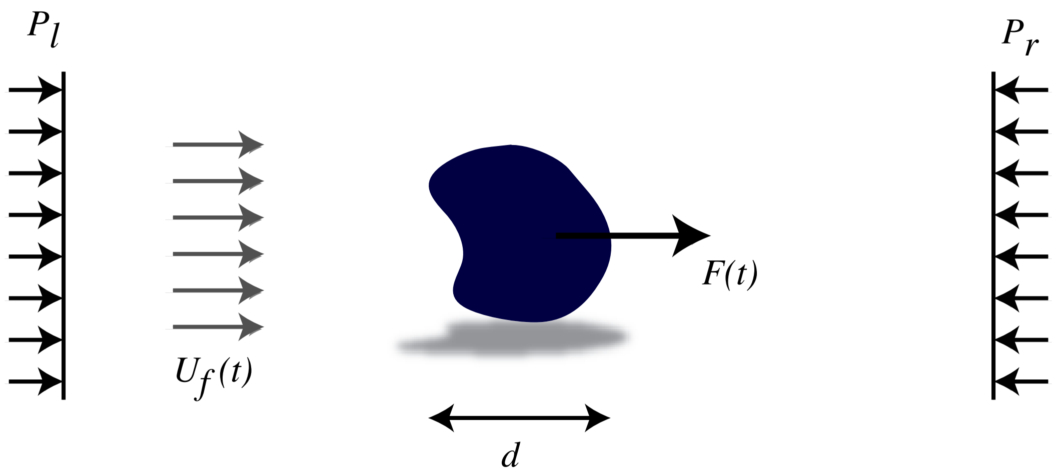


FIGURE 3. In order to accelerate or decelerate the fluid, a pressure gradient is necessary.

2.1. **Moving Fluid, Stationary Body.** In the case of a stationary body under the influence of an unsteady flow, an additional force term arises due in order to account for the change momentum in the fluid due to pressure gradients. In general, if the flow is variable, then a pressure gradient must be supplied to accelerate or decelerate the fluid: $(P_l - P_r)$.

For example, if fluid occupied the volume taken up by the body, ∇, then the force on that equivalent volume would equal the change in momentum of the fluid:

$$Force = \frac{d}{dt}\left[momentum\right] = \frac{d}{dt}\int\int\int \rho\ U_f\ d\nabla \tag{2.7}$$

We can assume that over the distance of the body, d, the flow is essentially unchanged. In this case we can rewrite equation 2.7 as follows:

$$Force = \left(\int \int \int d\nabla \right) \rho \frac{dU_f}{dt} \tag{2.8}$$

This force is referred to as the dynamic Archimedes force or Froude-Krylov force. For a stationary body in an unsteady flow we now have both an added mass term and this additional term:

$$F(t) = \rho \nabla \frac{dU_f}{dt} + m_a \frac{dU_f}{dt} \tag{2.9}$$

2.2. **Moving Body, Moving Fluid.** The case of a body translating in an unsteady fashion in an unsteady flow, the force must take into consideration the relatove force between the body and the fluid motion. Take convention to be both the body velocity and fluid velocity in the same direction (to the right, say), the force acting on the fluid can be written as:

$$F(t) = \rho \nabla \frac{dU_f}{dt} + m_a \left\{ \frac{dU_f}{dt} - \frac{dU_b}{dt} \right\} \tag{2.10}$$

2.3. **Moving Cylinder.** Let's look at the case of the moving cylinder in a quiescent fluid.

If a frictional force is not important compared to form drag then the energy in the fluid is represented by the energy inside and outside the wake.

$$E = Energyinfluid = Energyoutsidewake + Energyinwake = \frac{1}{2} m_a U^2 + (|U\tau|)\ h(\frac{1}{2}\rho u^2) \tag{2.11}$$

The quantity $(|U\tau|)$ is the distance travelled in time $t = \tau$, the volume is $(|U\tau|)\ h$ and h the wake width can be written as the cylinder diameter times the coefficient of drag, $h = C_d\ h$.

The drag force can be determined from the power in the wake, since it is equivalent to the time rate of change of fluid energy.

$$Power = D\ U = \frac{dE}{dt}(\tau = 0) \tag{2.12}$$

We can thus write the drag force as

$$D = \frac{1}{U}\frac{dE}{dt}(\tau = 0) = m_a \frac{dU}{dt} + \frac{1}{2}\rho h U^2 = m_a \frac{dU}{dt} + \frac{1}{2}\rho\ C_d\ d\ U\ |U| \tag{2.13}$$

When a body is motionless, but the fluid moves with velocity $U_f(t)$ we can write the force as

$$F(t) = (\rho\nabla + m_a)\frac{dU_f}{dt} + \frac{1}{2}\rho C_d\ d\ U_f\ |U_f|. \tag{2.14}$$

This equation is the familiar form of Morrison's Equation. This can be re-written if the body and the fluid are both in motion:

$$F(t) = \rho\nabla\frac{dU_f}{dt} + m_a\frac{d}{dt}(U_f - U_b) + \frac{1}{2}\rho C_d\ d\ (U_f - U_b)\ |U_f - U_b|. \tag{2.15}$$

3. Morrison's Equation in Waves

By the unsteady nature of water waves, Morrison's equation is quite applicable for determining the force on structures in waves. However we must assume that the presence of the structure does not influence the wave train (ie. there is no wave diffraction at the structure). In order for this assumption to be valid the wavelength of the waves must be larger than the body diameter.

Take the force in the x-direction on the structure in figure 4 per unit length (height) of the cylinder:

$$dF_x(t,z) = \left[\rho\frac{\pi}{4}d^2 + m_a\right]\frac{du}{dt}dz + \frac{1}{2}\rho C_D\ d\ u\ |u|\ dz \tag{3.1}$$

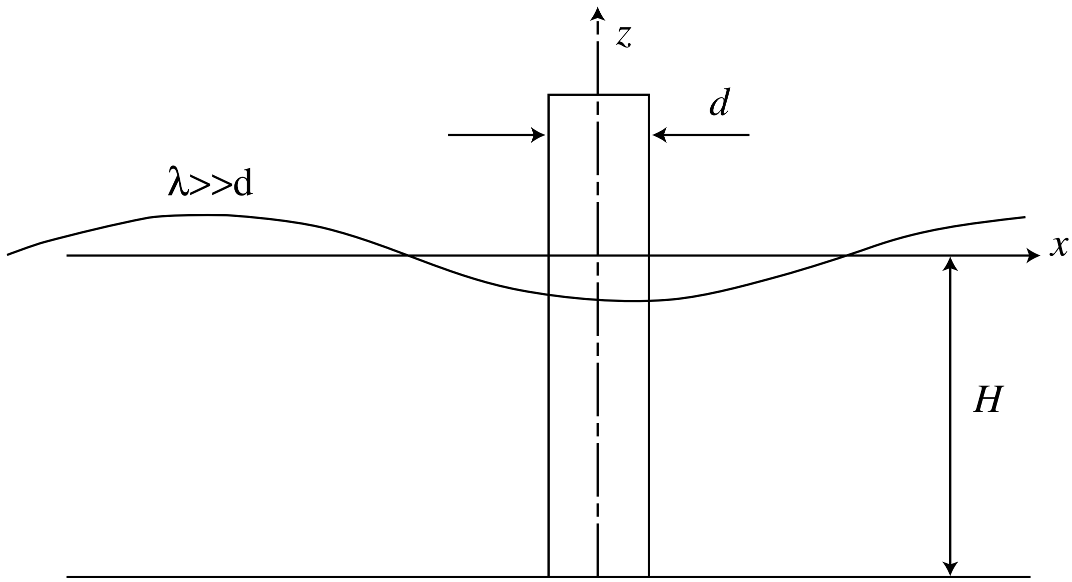


FIGURE 4. Morrison's equation can be used to formulate the force on a cylinder in waves with a wavelength much larger than the body diameter.

Recall from basic wave theory that the fluid velocity and acceleration, found from the wave potential, can be written at $x = 0$ as

$$u(x = 0, z, t) = a\omega \, \frac{\cosh \; k(z + H)}{\sinh \; kH} \cos\theta \tag{3.2}$$

$$\frac{du}{dt}(x = 0, z, t) = -a\omega^2 \, \frac{\cosh \; k(z + H)}{\sinh \; kH} \sin\theta, \tag{3.3}$$

where $\theta = (\omega t - kx)|_{x=0}$. By taking $x = 0$ to be at the center of the cylinder, and still assuming that the flow is essentially unchanged over the diameter of the body, this equation is simplified significantly.

For the purposes of design, we are most interested in the maximum possible forces on the structure. At $x = 0$ we can rewrite Morrison's equation in terms of the inertial and drag components:

$$F_x(t) = -F_I \; \sin\omega t + F_D \; \cos\omega t|\cos\omega t|. \tag{3.4}$$

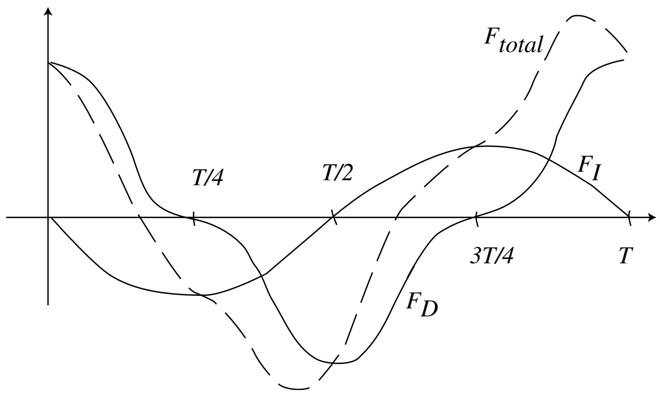


FIGURE 5. PLot of the inertial and drag components of morrison's equation.

Looking at equation 3.1 and the respective maxima for velocity and acceleration, we can see that a maximum occurs in the interval $\frac{3T}{4} \leq t < T$, where $\cos\omega t$ is positive (see figure 5). To find the maximum, take the first derivative of F_x at time t_o:

$$\frac{dF_x}{dt}(t_o) = 0 \tag{3.5}$$

$$F_I \cos\omega t_o = -F_D 2 \cos\omega t_o \sin\omega t_o \tag{3.6}$$

In order for this relationship to hold there are two options:

- (a.) $\cos(\omega t_o) = 0$ if $\frac{F_I}{2F_D} > 1$
- (b.) $\sin(\omega t_o) = -\frac{F_I}{2F_D}$ if $\frac{F_I}{2F_D} \leq 1$

Here we can see for case (a.) that at $t = T$ (one wave cycle period), the maximum force per unit length, F_{max}, results:

$$F_{max} = F_I \tag{3.7}$$

if $F_I > 2F_D$. The total force is found by integrating over the cylinder height.

(3.8)
$$F_x(t) = \int_{-H}^{0} dF_x(t,z) = -F_I \sin\theta + F_D \cos\theta |\cos\theta|$$

and plugging in the wave velocity and acceleration.

$$F_I = \left(\rho \tfrac{\pi}{4} d^2 + m_a\right) \tfrac{a\omega^2}{k} \text{ and } F_D = \tfrac{1}{2}\rho C_D \ d \ \tfrac{a^2\omega^2}{2k} f_2$$

where

$$f_2 = \tfrac{\sinh(2kH)-2kH}{\cosh(2KH)-1}$$

For case (b.) the maximum force can be written in terms of the inertial and drag force coefficients.

(3.9)
$$F_{max} = -F_I \sin\omega t_o + F_D \cos^2\omega t_o = -F_I\left(-\frac{F_I}{2F_D}\right) + F_D\left(1 - \frac{F_I^2}{4F_D^2}\right) = F_D + \frac{F_I^2}{4F_D}$$

for $F_I < 2F_D$. **Further analysis reveals that the drag force must become at lest equal to $\frac{1}{2}F_I$ *before* it affects the maximum force.** Considering the total force (force per unit length integrated from $-H$ to 0) we can find the ratio of the drag to the inertial forcing:

(3.10)
$$\frac{F_D}{F_I} = \frac{C_D}{\pi C_m}\ \frac{h}{d}\ \frac{\sinh(2kH)+2kH}{4\ \sinh^2(kH)}$$

When the drag and inertial forces are equal, given the coefficient of mass and drag are $C_D \simeq 1$ and $C_m \simeq 2$, the wave height ($h = 2a$) can be compared with the cylinder diameter to determine asymptotic solutions:

(3.11)
$$\frac{h}{d} = \frac{\pi C_m}{C_D}\ \frac{4\ \sinh^2(kH)}{\sinh(2kH)+2kH}$$

As $\frac{H}{\lambda} \to 0$ we approach the shallow water case and $\sinh(2kH) \simeq 2kH$ so

$$\frac{h}{d} \simeq (2\pi)^2 \ \frac{H}{\lambda} \tag{3.12}$$

As $\frac{H}{\lambda} \to \inf$ we approach the deep water case and $\sinh(2kH) \simeq e^k H$ so

$$\frac{h}{d} \simeq 4\pi \tag{3.13}$$

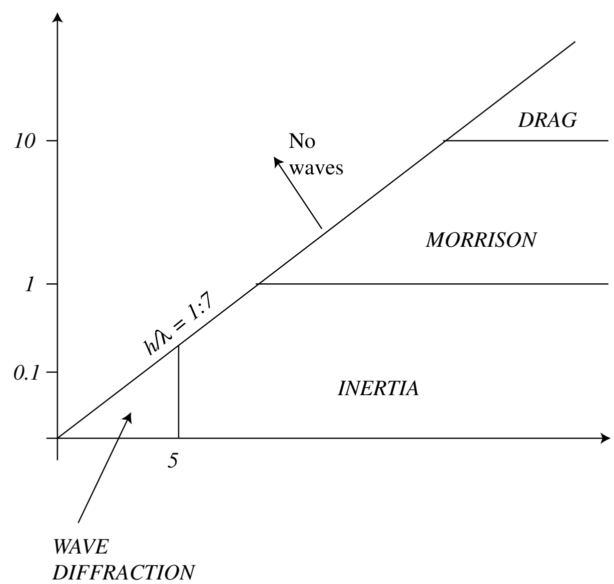


FIGURE 6. Morrison's Equation holds for waves with height to structure diameter (h/d) between 1 and 10.