13.012 Hydrodynamics for Ocean Engineers

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Archimedes's Principle and Static Stability

"Any object, wholly or partly immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

"The apparent loss in weight of a body immersed in a fluid is equal to the weight of the displaced fluid."

I. Archimedes's Principle:

The force on a body due to pressure alone (in the absence of viscous forces)

$$\vec{F} = \iint_S p\,\hat{n}\,ds \tag{3.1}$$

where pressure is a function of depth below the free surface:

$$p(z) = -\rho g z\,. \tag{3.2}$$

$$\vec{F} = \iint_S p\;\hat{n}\,ds = -\rho g \iint_S z\;\hat{n}\,ds \tag{3.3}$$

By Calculus the surface integral can be converted into a volume integral (Gauss's Theorem/Divergence Theorem):

$$\iint_S \vec{G}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{G}\;dV \tag{3.4}$$

Thus equation becomes:

$$\vec{F} = \iint_S p\,\hat{n}\,ds = -\rho g\iint_S z\,\hat{n}\,ds = \rho g \iiint_V dV = \rho g V\,\hat{k} \tag{3.5}$$

We can see now that the buoyancy force acts to counterbalance the displaced volume of fluid. For a half submerged body the area of the water plane must be accounted for in the integration.

II. Moment on a body (Ideal Fluid)

The moment on a submerged body follows directly from structural mechanics or dynamics methodologies.

$$\vec{M} = \iint_S p(\vec{x}\times\hat{n})\, ds \equiv (M_1, M_2, M_3) = (M_x, M_y, M_z) \qquad (3.6)$$

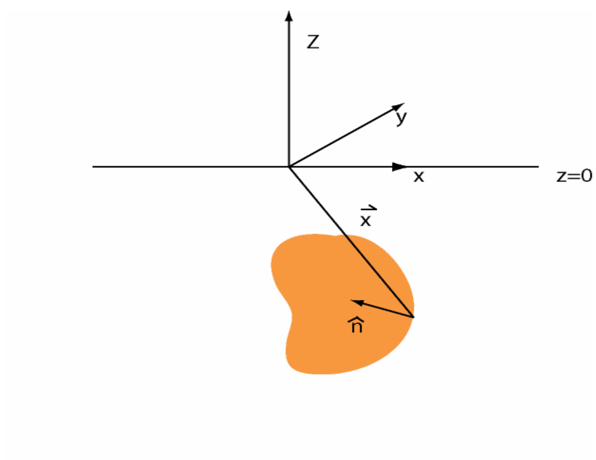


Figure 1: *x, y, z* coordinate reference frame.

Recall from vector calculus:

$$\vec{x}\times\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{matrix} \hat{i}(yn_3 - zn_2) \\ -\hat{j}(xn_3 - zn_1) \\ +\hat{k}(xn_2 - yn_1) \end{matrix} \qquad (3.7)$$

So we get the moments as follows:

$$M_1 = \iint_s p(yn_3 - zn_2)\, ds \qquad (3.8)$$

M_1 is taken about the x-axis. We can also calculate in a similar fashion: M_2 about the y-axis and M_3 about the z-axis.

III. Center of Buoyancy

Calculating the center of buoyancy, it is first necessary to find the center of area.

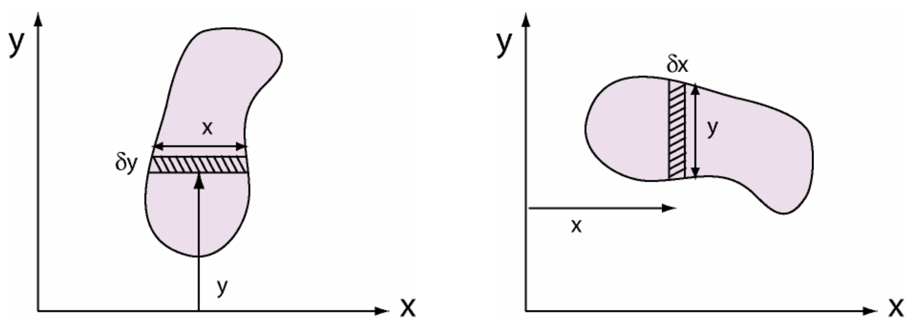


Figure 2: Determining the center of area in *x* and *y*

The total area, *A*, can be found by integrating small sections of the body in either direction:

$$A = \int y\, dx = \int x\, dy \tag{3.9}$$

When calculating a ships area, an approximate area is often used since it is hard to define a mathematical formula for the ships geometry.

The first moment of the area can be found integrating the height, y_1, of a section located at some x position along the x-axis for M_{yy} and integrating the height, x_1, of a section located at some y position along the y-axis for M_{xx}:

$$M_{yy} = \int y_1\, x\, dx \tag{3.10}$$

$$M_{xx} = \int x_1\, y\, dy \tag{3.11}$$

The coordinates for the center of area are then found using equation (3.10), (3.11) and the formula for the object's area, (3.9):

$$\bar{x} = \frac{M_{yy}}{A} = \frac{1}{A}\int x\, y_1 \, dx \tag{3.12}$$

$$\bar{y} = \frac{M_{xx}}{A} = \frac{1}{A}\int y\, x_1 \, dy \tag{3.13}$$

Center of buoyancy ($\bar{x}, \bar{y}$) is the point at which the buoyancy force acts on the body and is equivalently the geometric center of the submerged portion of the hull.

IV. Static Equilibrium: Gravity and buoyancy balance

A body at rest is considered to be in equilibrium.

The mass of the object (vessel) is equal to the mass of an equivalent volume of water (displacement). When a vessel is in static equilibrium, the center of gravity and buoyancy act on the same vertical line. Buoyancy acts perpendicular to the water line (upwards) at the center of buoyancy.

Center of gravity usually lies amidships and near to the water line.

Center of buoyancy changes dependant on the volume of the submerged portion of the vessel. As a ship rolls, pitches, etc., the center of buoyancy moves according to the "new" shape of the submerged block. Take for example a container ship: as containers are loaded the weight of the ship increases and it submerges further into the water. With the increased draft the center of buoyancy shifts.

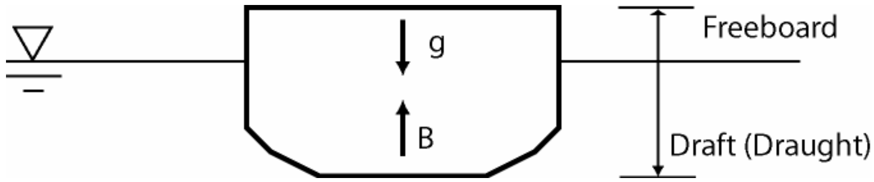


Figure 3: *A vessel in static equilibrium with the centers of gravity and buoyancy aligned in the same vertical line.*

V. Righting moments: The moment required/available to right a vessel.

An object disturbed from a static position that returns to its original position is considered statically stable or in *static equilibrium*. Such vessels are "self-righting". However if the vessel is unable to return to its equilibrium position and continues to turn over (capsize) it is considered unstable. It a vessel is in equilibrium at any position then it is neutrally stable (take for example a simple cylinder on its side – turn it to any angle and it will stay).

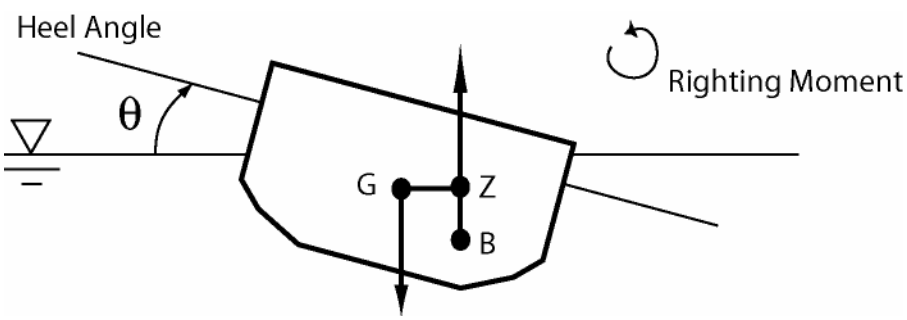


Figure 4: A statically stable vessel with a positive righting arm.

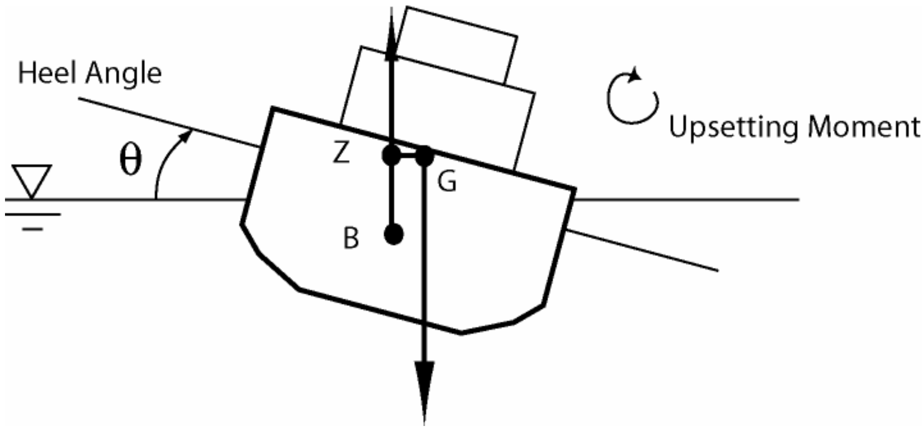


Figure 5: Statically unstable vessel with a negative righting arm.

The righting moment is equivalent to the Displacement of the vessel times the righting arm:

$$RM = W \times \overline{GZ} \quad (3.14)$$

VI. Small rotation about the y-axis equation for stability:

Sum of the torques about the center of rotation must be zero (conservation of angular momentum!)

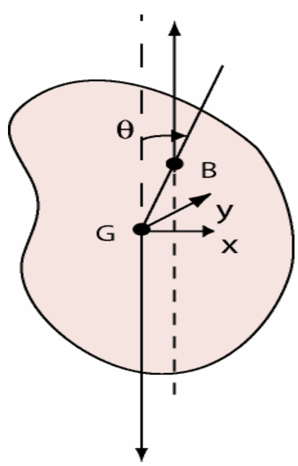


Figure 6: Arbitrary body with center of gravity, *G*, center of buoyancy, *B*, and angle of rotation, *θ*.

We can write the equation of motion in terms of the rotational displacement, θ, and rotational acceleration, $\ddot{\theta}$, for small angles:

$$I\ddot{\theta} + gl\,\theta = 0 \tag{3.15}$$

where $l = |GB|$, and I is the moment of inertia per unit mass. This is the equation of simple harmonic motion similar to a pendulum. The motion is in stable equilibrium when $l > 0$ and unstable when $l < 0$.

VII. Metacenter: A ships metacenter is the intersection of the two lines of action of the buoyancy force.

What does this mean?

Basically it means that as a ship rolls (heels) through an angle, θ, the center of buoyancy, B, shifts along a semi-elliptical path (depending on geometry). Since the buoyancy always acts vertically, the two lines of action for two different heel angles must cross at a point. This point is the metacenter of the ship, denoted by M in figure 7.

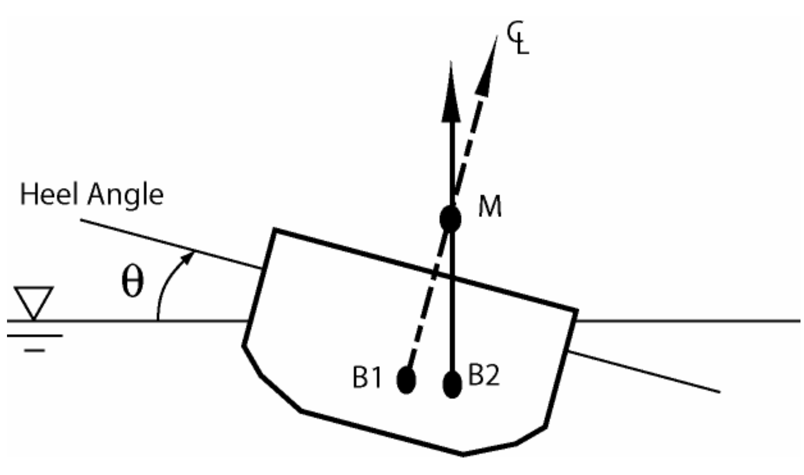


Figure 7: The metacenter is the intersection of two distinct lines of action of the buoyancy force.

When on at an angle of roll (heel) the center of buoyancy and center of gravity no longer act on the same vertical line of action.

Transverse Metacenter: Roll/Heel

Longitudinal Metacenter: Pitch/Trim

The concept of the metacenter is only really valid for small angles of motion.

Metacentric height is the distance measured from the metacenter to the center of gravity: $\overline{GM}$. If the metacentric height is large, then the vessel is considered to be "stiff" in roll – indicating that there will be a large righting moment as a result of small roll angles. In contrast, if the metacentric height is small then the ship rolls slowly due to a smaller righting arm.

How do you determine $\overline{GM}$?

First let's define a few pertinent quantities:

$\overline{KB}$ = Center of Buoyancy above the Keel
$\overline{KM}$ = Height of Metacenter above Keel
$\overline{BM}$ = Distance between center of Buoyancy and Metacenter
$\overline{KG}$ = Center of Gravity above the Keel
$\overline{GM}$ = Metacentric height

Take for example a ship in roll, we can find $\overline{GM}$ by simply by knowing a few simple things:

1) Find $\overline{KB}$. As a rule of thumb for first estimates this can be taken as about 52% of the draft. Numerical analysis can be used to improve this guess.

2) Find $\overline{BM}$, the metacentric radius, using the transverse moment of inertia of the ships water plane about its own centerline and the vessel's displaced volume:

$$\overline{BM} = \frac{I}{Volume} \tag{3.16}$$

3) Determine $\overline{KG}$, or the vertical center of all weights on the ship taking into account the empty ship plus all deadweight items. This can change so $\overline{KG}$ is usually only an estimate. Over-estimating leads to higher safety to some extent.

4) Calculate $\overline{GM}$:

$$\overline{KM} = \overline{KB} + \overline{BM} \tag{3.17}$$

$$\overline{GM} = \overline{KM} - \overline{KG} \tag{3.18}$$

We could also determine the Metacentric height of a vessel by attempting a simple experiment. Take a vessel floating on the free surface and place a weight directly over the center of gravity. The ship will settle slightly deeper into the water but not roll. Now move the weight some distance out from the centerline and measure the angle of roll:

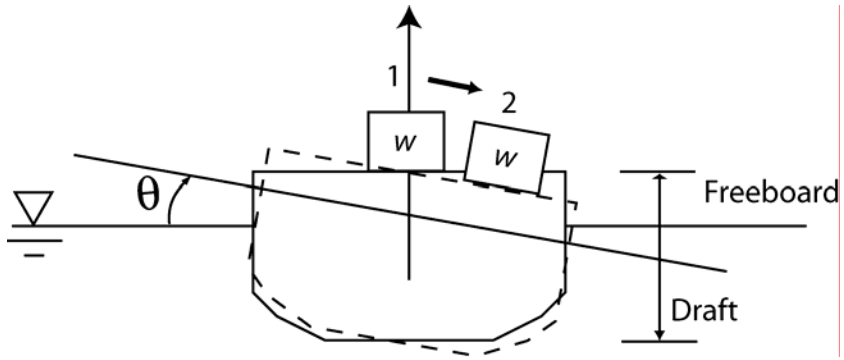


Figure 8: Ship rolling due to cargo shifting.

The metacentric height can be determined using the simple formula for small angles ($\sin\theta \approx \tan\theta$):

$$\overline{GM} = \frac{wd}{W\tan\theta} \tag{3.19}$$

where *w* is the mass of the weight (lbs), *d* is the distance (inches) the weight is moved, *W* is the displacement of the vessel (lbs) and $\tan\theta$ is the tangent of the roll (list) angle. (NB. A ship is said to be listing when it is leaning to one side).

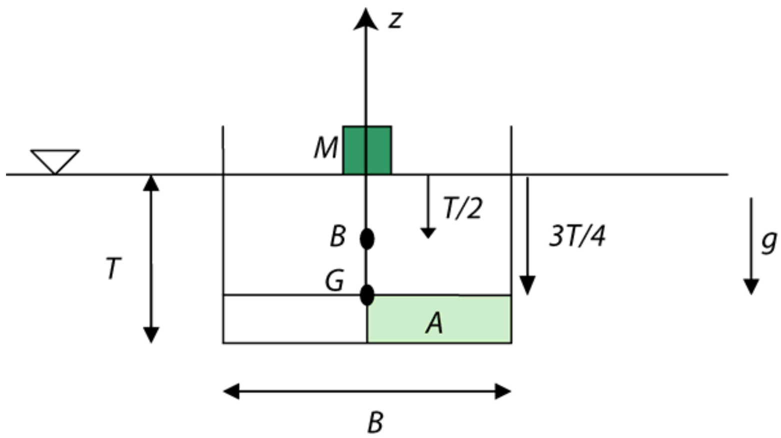
In designing ships naval architects want a reasonable value of $\overline{GM}$ of approximately 3-5% of the beam of the ship. Smaller values do not leave much margin for error in loading a ship. Larger values can make a ship excessively stable.

When we refer to a ship's stability we usually consider it *stable with respect to the water's surface*. Such that a ship with high stability will follow every wave profile that it encounters. This can lead to dangerous on-board conditions not to mention sea-sickness.

US Coast Guard and other regulators specify what acceptable values of $\overline{GM}$ are for safe ship operation, giving upper and lower thresholds for this value.

Example Problem:

You are called on the scene when a double hull tanker unfortunately runs into a migrating whale shark (while very gentle these sharks are very large). This collision has created a small crack in the outer hull allowing water to flood the lower compartment. A pump with mass, *M*, is on deck at the centerline of the vessel. A cross sectional view of the vessel is shown below:



1) Determine the vertical position of the metacentric height.
2) The empty compartment A is flooded with water and the tanker assumes a small heel angle, δ. Determine the angle in terms of the given quantities.
3) In order to stabilize the tanker to a level position the captain decides to shift the pump horizontally on deck and asks you to determine the necessary horizontal shift for the heel angle to vanish.
4) In order not to run aground coming into port the captain must know the new draft of the vessel following the adjustment in part c and again asks you to calculate it for him.

NB: Whale Shark (*Rhincodon typus*)

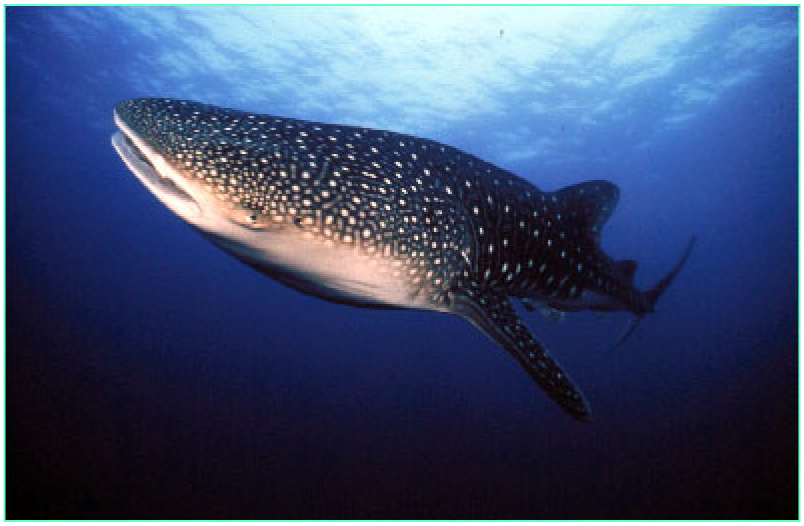


Image from http://www.boattalk.com/sharks/whaleshark.htm

Whale sharks will grow to over 12 metres in length, which is about the size of a large bus. These gentle ocean giants are often confused with whales because of their large size and feeding habits. They are, however, sharks, albeit the least fearsome of this group, and their closest relatives are the nurse and wobbegong sharks.

Whale sharks are not aggressive, and like the second largest of all sharks - the slightly smaller basking shark - cruise the oceans feeding on concentrations of zooplankton, small fish and squid. The whale shark's mouth contains 300 rows of tiny teeth, but ironically, they neither chew nor bite their food. Instead, the sharks use a fine mesh of rakers attached to their gills to strain food from the water. These rakers are functionally similar to the baleen plates possessed by many whales.

Biologists have speculated that whale sharks feed by literally vacuuming food from the water. However, researchers at Ningaloo have observed that the sharks usually feed by actively swimming through a mass of zooplankton or small fish with their mouths wide open. Whale sharks have also been observed to hang vertically in the water and feed by sucking water into their mouths. (http://www.fish.wa.gov.au/rec/broc/fishcard/whaleshk.html)