13.012 Hydrodynamics for Ocean Engineers

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Fall 2004

Fluid Forces on Bodies

1. Steady Flow

In order to design offshore structures, surface vessels and underwater vehicles, an understanding of the basic fluid forces acting on a body is needed. In the case of steady viscous flow, these forces are straightforward. *Lift* force, perpendicular to the velocity, and *Drag* force, inline with the flow, can be calculated based on the fluid velocity, U, force coefficients, C_D and C_L, the object's dimensions or area, A, and fluid density, ρ.

$$F_{Drag} = \frac{1}{2}\rho U^2 A\, C_D \qquad (1.1)$$

$$F_{Lift} = \frac{1}{2}\rho U^2 A\, C_L \qquad (1.2)$$

These equations can also be used in a quiescent (stationary) fluid for a steady translating body, where U is the body velocity instead of the fluid velocity.

2. Unsteady Motion

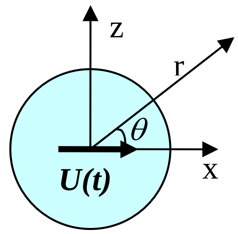
Beyond steady flow, especially in the presence of free surface waves, we must consider unsteady, time dependent motions of both the fluid and the body and the fluid inertial forces arise, adding to the total forcing on a body.

Take the case of an unsteady moving body, $U_b(t)$, in an unbounded inviscid, irrotational fluid ($\mu = 0$) with zero velocity, $U_f = 0$. The time-dependent force on the body is directly proportional to the body acceleration:

$$F(t) = -m_a \frac{dU_b(t)}{dt} \qquad (1.3)$$

where m_a, is the system added mass, depends on the body geometry and direction of motion. This is an *added* inertial force or *added mass* force on the body. By comparison, in an inviscid steady flow, by D'Alembert's Paradox, the force on the body would be zero.

Unsteady Moving Body Stationary Fluid: Force on a sphere (radius *a*) accelerating in an unbounded quiescent (non-moving) fluid. $U = U(t)$ is the unsteady body velocity.



The Kinematic Boundary Condition on the sphere, guaranteeing no fluid flow through the body surface, is

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = U(t)\cos\theta. \qquad (1.4)$$

The potential function for a moving sphere with no free stream (still fluid) is simply

$$\phi = -U(t)\frac{a^3}{2r^2}\cos\theta. \qquad (1.5)$$

You can double check this solution for the velocity potential by substituting ϕ into the Kinematic Boundary Condtion (eq. (1.4)) to make sure this potential works at the boundary of the sphere.

The hydrodynamic force on the body due to the unsteady motion of the sphere is given as a surface integral of pressure around the body. Pressure formulation comes from the unsteady form of Bernoulli. Force in the x-direction is

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \qquad (1.6)$$

Since we want the force acting on the body we need the velocity components on the sphere surface ($r = a$).

In spherical coordinates the velocity is found by taking the gradient of the potential function as follows:

$$\vec{V} = \nabla\phi = \left(V_r, V_\theta, V_\varphi\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) \tag{1.7}$$

such that the gradient of $\phi(t)$ for a moving sphere is

$$\nabla\phi\big|_{r=a} = \left(U(t)\cos\theta, \frac{1}{2}U(t)\sin\theta, 0\right). \tag{1.8}$$

To evaluate the pressure around the sphere we need the magnitude of the velocity $\left|\nabla\phi\right|^2$ on the surface ($r = a$):

$$\left(\left|\nabla\phi\right|^2\right)\Big|_{r=a} = U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta;\ \hat{n} = -\hat{e}_r;\ n_x = -\cos\theta \tag{1.9}$$

Next, the time derivative of the velocity potential, evaluated at the sphere surface, is

$$\left.\frac{\partial\phi}{\partial t}\right|_{r=a} = -\dot{U}\left(t\right)\frac{a^3}{2r^2}\cos\theta\Big|_{r=a} = -\frac{1}{2}\dot{U}\left(t\right)a\cos\theta \tag{1.10}$$

and the surface integral can be re-written in spherical coordinates as

$$\iint_b dS = \int_0^\pi (a\ d\theta)(2\pi a\sin\theta) \tag{1.11}$$

Substituting (1.9), (1.10), and (1.11) into (1.6) we can solve for *Fx*, the added mass force on a spherical body moving with an unsteady acceleration:

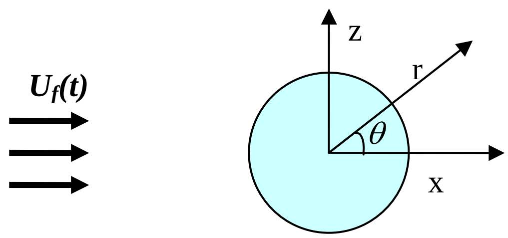
$$F_x = -\dot{U}(t)\left(\underbrace{\rho\frac{2}{3}\pi a^3}_{unit=Mass}\right) \tag{1.12}$$

The volume of a sphere is $\forall_s = \frac{4}{3}\pi a^3$ thus equation (1.12) is simply

$$\boxed{F_x = -\dot{U}(t)\left(\tfrac{1}{2}\rho\forall_s\right)} \tag{1.13}$$

where $m_a = \frac{1}{2}\rho\forall_s$ is the added mass in the system.

Unsteady Moving Fluid Stationary Body: Force on a sphere (radius *a*) in an unbounded unsteady moving fluid. $U_f = U_f(t)$ is the unsteady fluid velocity.



The corresponding potential function is simply:

$$\phi(r,\theta,t) = U_f(t)\left(r + \frac{a^3}{2r^2}\right)\cos\theta \tag{1.14}$$

The velocity, at the body on r=a, can only be tangential to the body due to the kinematic boundary condition ($\text{KBC} \Rightarrow \text{V}_\text{r} = 0\big|_{r=a}$) :

$$\vec{V} = \nabla\phi\big|_{r=a} = \left(0, -\tfrac{3}{2}U_f\sin\theta, 0\right) \tag{1.15}$$

The time derivative of the potential is

$$\left.\frac{\partial\phi}{\partial t}\right|_{r=a} = \dot{U}_f\,\frac{3a}{2}\cos\theta \tag{1.16}$$

The magnitude of the velocity is simply:

$$\frac{1}{2}\left|\nabla^2\phi\right| = \frac{9}{8}U^2\sin^2\theta \tag{1.17}$$

Again, using (1.16), (1.17), (1.11) in the formulation for F_x, from equation (1.6), we get the horizontal force on the body:

$$F_x = (-\rho)\left(2\pi a^2\right)\int_0^\pi -\cos\theta\sin\theta\left[\dot{U}_f \tfrac{3a}{2}\cos\theta + \tfrac{9}{8}U_f^2\sin^2\theta\right]d\theta \tag{1.18}$$

$$F_x = 3\rho\pi a^3 \dot{U}_f \int_0^\pi \cos^2\theta\sin\theta\, d\theta + \tfrac{9}{4}\rho\pi a^2 U_f^2 \int_0^\pi \sin^3\theta\cos\theta\, d\theta \tag{1.19}$$

By parts the right most term in the integral in (1.19) reduces to zero, and the integral from left term is simply:

$$\int_0^\pi \cos^2\theta\sin\theta\, d\theta = \frac{2}{3} \tag{1.20}$$

so we are left with a force:

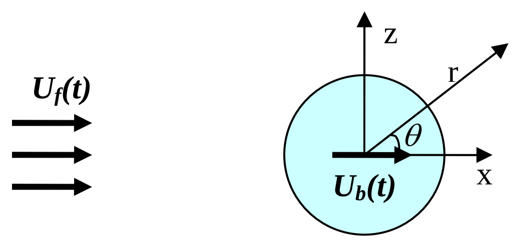
$$F_x = 3\rho\pi a^3 \dot{U}_f \cdot \frac{2}{3} = 2\rho\pi a^3 \dot{U}_f \tag{1.21}$$

This can be rewritten in terms of the sphere volume, $\forall_s = \frac{4}{3}\pi a^3$, and added mass, $m_a = \frac{1}{2}\rho\forall_s$, from (1.13) as follows

$$\boxed{F_x = \dot{U}_f\left(\rho\forall + m_a\right)} \tag{1.22}$$

The non-added mass term of equation (1.22) is due to the pressure gradient necessary to accelerate the fluid around the sphere. This is like a buoyancy effect.

Unsteady Moving Fluid; Unsteady Moving Body: Force on a moving sphere (radius *a*) in an unbounded moving fluid. $U_f = U_f(t)$ is the unsteady fluid velocity and $U_b = U_b(t)$ is the body velocity.



The case of the unsteady moving body and fluid can be determined by combining the results from the previous two cases.

$$\boxed{F_x = -\dot{U}_b\left(\tfrac{1}{2}\rho\forall_s\right)}$$
Moving Body
Still Fluid

$$\boxed{F_x = \dot{U}_f\left(\rho\forall + m_a\right)}$$
Moving Fluid
Still Body

Moving Body, Moving Fluid:

$$F_x = -\dot{U}_b\left(m_a\right) + \dot{U}_f\left(\rho\forall + m_a\right) = \dot{U}_f\rho\forall + m_a\left(\dot{U}_f - \dot{U}_b\right) \qquad (1.23)$$

So now, all we have to do is find the added mass!

3. Added Mass

For the case of unsteady motion of bodies underwater or unsteady flow around objects, we must consider the additional effect (force) acting on the structure when formulating the system equation.

Most floating structures can be modeled, for small motions and linear behavior, by a system equation with the basic form similar to a typical mass-spring-dashpot system described by the following equation:

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{1.24}$$

where *m* is the system mass, *b* is the linear damping coefficient, *k* is the spring coefficient, *f(t)* is the force acting on the mass, and *x* is the displacement of the mass. The natural frequency ω of the system is simply

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{1.25}$$

In a physical sense, this added mass is the weight added to a system due to the fact that an accelerating or decelerating body (ie. unsteady motion: $dU/dt \neq 0$) must move some volume of surrounding fluid with it as it moves. The added mass force opposes the motion and can be factored into the system equation as follows:

$$m\ddot{x} + b\dot{x} + kx = f(t) - m_a\ddot{x} \tag{1.26}$$

where m_a is the added mass. Reordering the terms the system equation becomes:

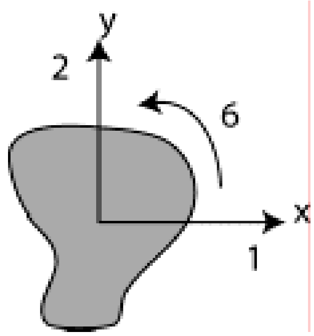
$$\left(m + m_a\right)\ddot{x} + b\dot{x} + kx = f(t) \tag{1.27}$$

From here we can treat this again as a simple spring-mass-dashpot system with a new mass $m' = m + m_a$ such that the natural frequency of the system is now

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k}{m + m_a}} \tag{1.28}$$

It is important in ocean engineering to consider floating vessels or platforms motions in more than one direction. Added mass forces can arise in one direction due to motion in a different direction, and thus we can end up with a 6 x 6 matrix of added mass coefficients.

Looking simply at a body in two-dimensions we can have linear motion in two directions and rotational motion in one direction. (Think of these coordinates as if you were looking down on a ship.)



Two dimensional motion with axis (x,y) fixed on the body. 1: Surge, 2: Sway, 6: Yaw

The unsteady forces on the body in the three directions are:

$$-F_1 = m_{11}\frac{du_1}{dt} + m_{12}\frac{du_2}{dt} + m_{16}\frac{du_6}{dt} \tag{1.29}$$

$$-F_2 = m_{21}\frac{du_1}{dt} + m_{22}\frac{du_2}{dt} + m_{26}\frac{du_6}{dt} \tag{1.30}$$

$$-F_6 = m_{61}\frac{du_1}{dt} + m_{62}\frac{du_2}{dt} + m_{66}\frac{du_6}{dt} \tag{1.31}$$

Where F_1, F_2, and F_6, are the surge (x-) force, sway (y-) force and yaw moments respectively. It is common practice in Ocean Engineering and Naval Architecture to write the moments for roll, pitch, and yaw as F_4, F_5, and F_6 and the angular motions in these directions as X_4, X_5, and X_6.

This set of equations, (1.29)-(1.31), can be written in matrix form, $\underline{F} = [M]\underline{\dot{u}}$,

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ m_{61} & m_{62} & m_{66} \end{bmatrix} \begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \\ \frac{du_6}{dt} \end{pmatrix} \tag{1.32}$$

Considering <u>all</u> six degrees of freedom the Force Matrix is:

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} \tag{1.33}$$

We will often abbreviate how we write the Force matrix given in (1.33) using tensor notation.

The force vector is written as

$$\underline{F} = F_i \text{, where } i = \underbrace{1,2,3}_{\substack{Linear\\Forces}},\ \underbrace{4,5,6}_{Moments}, \tag{1.34}$$

the acceleration vector as

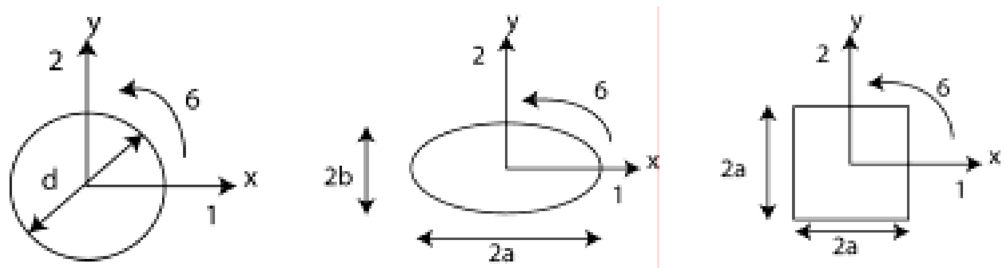
$$\dot{u}_i = [u_1, u_2, u_3, u_4, u_5, u_6], \tag{1.35}$$

and the added mass matrix $[m_a]$ as

$$m_{ij} \text{ where } i, j = 1,2,3,4,5,6. \tag{1.36}$$

A good way to think of the added mass components, m_{ij}, is to think of each term as mass associated with a force on the body in the i^{th} direction due to a *unit* acceleration in the j^{th} direction.

For symmetric geometries the added mass tensor simplifies significantly. For example, figure 2 shows added mass values for a circle, ellipse, and square. In the case of the circle and square, movement in the 1 and 2 directions yields similar geometry and identical added mass coefficients ($m_{11} = m_{22}$).



Circle	Ellipse	Square
$m_{11} = m_{22} = \rho d^2$	$m_{11} = \rho \pi b^2$	$m_{11} = m_{22} = 1.51 \pi \rho a^2$
$m_{66} = 0$	$m_{22} = \rho \pi a^2$	$m_{66} = 0.234 \pi \rho a^4$
	$m_{66} = \rho \left(a^2 - b^2 \right)^2$	

Two dimensional added mass coefficients for a circle, ellipse, and square in 1: Surge, 2: Sway, 6: Yaw

Using these coefficients and those tabulated in Newman's *Marine Hydrodynamics* on p.145 we can determine the added mass forces quite simply.

In three-dimensions, for a sphere (by symmetry):

$$m_{11} = m_{22} = m_{33} = \frac{1}{2} \rho \forall = m_A \qquad (1.37)$$

ALL OTHER m_{ij} TERMS ARE ZERO ($i \neq j$).

General 6 DOF forces and moments on a Rigid body moving in a fluid:

Velocities:

$$Translation\,Velocity: \vec{U}(t) = (U_1, U_2, U_3) \tag{1.38}$$

$$Rotational\,Velocity: \vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6) \tag{1.39}$$

All rotation is taken with respect to Origin of the coordinate system (often placed at the center of gravity of the object for simplicity!).

Forces: (force in the j^{th} direction). ($i = 1,2,3,4,5,6$ and $j,k,l = 1,2,3$)

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li} \tag{1.40}$$

Moments: ($i = 1,2,3,4,5,6$ and $j,k,l = 1,2,3$)

$$M_j = -\dot{U}_i m_{j+3,i} - \varepsilon_{jkl} U_i \Omega_k m_{l+3,i} - \varepsilon_{jkl} U_k U_i m_{li} \tag{1.41}$$

Einstein's summation notation applies!

The alternating tensor ε_{jkl} is simply

$$\varepsilon_{jkl} = \begin{cases} 0; & \text{if any } j,k,l \text{ are equal} \\ 1; & \text{if } j,k,l \text{ are in cyclic order} \\ -1; & \text{if } j,k,l \text{ are in anti-cyclic order} \end{cases} \tag{1.42}$$

The full form of the force in the x-direction (F_1) is summed over all values of *i*:

$$\begin{aligned} \underbrace{F_1}_{j=1} = & -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6} \\ & - \underbrace{\varepsilon_{1kl} U_1 \Omega_k m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl} U_2 \Omega_k m_{l2}}_{i=2} - \underbrace{\varepsilon_{1kl} U_3 \Omega_k m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl} U_4 \Omega_k m_{l4}}_{i=4} \\ & - \underbrace{\varepsilon_{1kl} U_5 \Omega_k m_{l5}}_{i=5} - \underbrace{\varepsilon_{1kl} U_6 \Omega_k m_{l6}}_{i=6} \end{aligned} \tag{1.43}$$

for $k,l = 1,2,3$.

Next we can choose the index *k* to cycle through. It is helpful to note that the only terms where *k* plays a role, contain ε_{jkl}. Following the definition for ε_{jkl} given in (1.42) and since *j = 1*, all terms will be zero for *k = 1*. Therefore *k* can only take the value of 2 or 3:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$\underbrace{-\underbrace{\varepsilon_{12l} U_1 \Omega_2 m_{l1}}_{i=1} - \underbrace{\varepsilon_{12l} U_2 \Omega_2 m_{l2}}_{i=2} - \underbrace{\varepsilon_{12l} U_3 \Omega_2 m_{l3}}_{i=3} - \underbrace{\varepsilon_{12l} U_4 \Omega_2 m_{l4}}_{i=4} - \underbrace{\varepsilon_{12l} U_5 \Omega_2 m_{l5}}_{i=5} - \underbrace{\varepsilon_{12l} U_6 \Omega_2 m_{l6}}_{i=6}}_{k=2}$$

$$\underbrace{-\underbrace{\varepsilon_{13l} U_1 \Omega_3 m_{l1}}_{i=1} - \underbrace{\varepsilon_{13l} U_2 \Omega_3 m_{l2}}_{i=2} - \underbrace{\varepsilon_{13l} U_3 \Omega_3 m_{l3}}_{i=3} - \underbrace{\varepsilon_{13l} U_4 \Omega_3 m_{l4}}_{i=4} - \underbrace{\varepsilon_{13l} U_5 \Omega_3 m_{l5}}_{i=5} - \underbrace{\varepsilon_{13l} U_6 \Omega 3 m_{l6}}_{i=6}}_{k=3} \tag{1.44}$$

Finally we cycle through the index *l*. Again it is helpful to note that the only terms where *l* plays a role, contain ε_{jkl}. Following the definition for ε_{jkl} given in (1.42) and since *j = 1*, and *k = 2* or *3*, then all terms will be zero for *l = 1* and some zero for the case *l = 2* and others zero when *l =3*. Like before *l* can only take the value of 2 or 3 such that $l \neq k \neq j$:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$\underbrace{-\underbrace{\varepsilon_{123} U_1 \Omega_2 m_{31}}_{i=1} - \underbrace{\varepsilon_{123} U_2 \Omega_2 m_{32}}_{i=2} - \underbrace{\varepsilon_{123} U_3 \Omega_2 m_{33}}_{i=3} - \underbrace{\varepsilon_{123} U_4 \Omega_2 m_{34}}_{i=4} - \underbrace{\varepsilon_{123} U_5 \Omega_2 m_{35}}_{i=5} - \underbrace{\varepsilon_{123} U_6 \Omega_2 m_{36}}_{i=6}}_{k=2;\, l=3}$$

$$\underbrace{-\underbrace{\varepsilon_{132} U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} U_2 \Omega_3 m_{22}}_{i=2} - \underbrace{\varepsilon_{132} U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} U_4 \Omega_3 m_{24}}_{i=4} - \underbrace{\varepsilon_{132} U_5 \Omega_3 m_{25}}_{i=5} - \underbrace{\varepsilon_{132} U_6 \Omega 3 m_{26}}_{i=6}}_{k=3;\, l=2} \tag{1.45}$$

On the second row of the equation above, the indices of the alternating tensor, ε_{jkl}, are in cyclic order *jkl = 123* ($\varepsilon_{123} = +1$). In the third row, the indices are in anti (or reverse) cyclic order: $\varepsilon_{132} = -1$ where *jkl = 132.*

More than likely you will never have to write out all six force equations with all the terms as the velocity and acceleration of the body will be zero in certain directions. However for a full seakeeping analysis of a ship then one day you just might need to be able to determine all the forces!

Typical Example: For a body moving in the fluid with velocity

$$\vec{V} = (1,0,1,0,0,1) = (U_1, 0, U_3, 0, 0, U_6) = (U_1, 0, U_3, 0, 0, \Omega_3) \qquad (1.46)$$

and acceleration

$$\vec{a} = (1,0,0,0,0,1) = (\dot{U}_1, 0, 0, 0, 0, \dot{U}_6) \qquad (1.47)$$

we can find the force on the body in the X-direction. The force in the x-direction is F_1 so *j=1*.

First substitute "1" for every instance of *j* in equation (1.40) to get:

$$F_{j=1} = F_1 = -\dot{U}_i \, m_{i1} - \varepsilon_{1kl} \, U_i \Omega_k \, m_{li} \qquad (1.48)$$

Next we need to "cycle" through the possible values for *i* (*i = 1,2,3,4,5,6*). Looking at equation (1.48), it is clear that the only "i^{th}" accelerations that will matter are the non-zero ones from (1.47), thus $\dot{U}_1$ and $\dot{U}_6$, and the only "i^{th}" velocities to consider are for *i = 1,3,* and *6* [eqn (1.46)].

$$F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} - \underbrace{\varepsilon_{1kl} \, U_1 \Omega_k \, m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl} \, U_3 \Omega_k \, m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl} \, U_6 \Omega_k \, m_{l6}}_{i=6} \qquad (1.49)$$

Now look at the *k-index*: $(k \neq j \therefore k = 2,3)$ However $\Omega_2 = 0$ and $\Omega_3 \neq 0$ thus for $k = 2$ all associated terms will be zero, so we only have to deal with $k = 3$. Since $j = 1$ and $k = 3$ the only value left for *l*, that could result in non-zero terms, is 2.

$$\begin{aligned} F_1 = & -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} \\ & \underbrace{- \underbrace{\varepsilon_{132} \, U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} \, U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} \, U_6 \Omega_3 m_{26}}_{i=6}}_{k=3;\, l=2} \end{aligned} \qquad (1.50)$$

If the body in question was a simple, symmetrical sphere we could reduce this even further. Using the added mass values from (1.37) and trusting that the off-diagonal added mass terms are zero (just for the sphere), the force in the x-direction on a sphere, given (1.46) and (1.47), is

$$\boxed{F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1}} \qquad (1.51)$$

