13.012 Hydrodynamics for Ocean Engineers
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Determining Added Mass Using Slender Body Theory

1. Slender Bodies and Added Mass

To formulate the added mass of a system such as a ship or submarine that can be modeled as a *slender body*, we first need the two-dimensional sectional added mass coefficients. We will consider a *slender body* to have a characteristic length in one direction that is considerably longer than its length in the other two directions. For these slender bodies we can use known 2D coefficients to find the unknown 3D added mass coefficient for the body.

The added mass *force* acting on the body due to unsteady motion is

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li} \tag{7.1}$$

where m_{ij} is the added mass in the ith direction due to a unit acceleration in the jth direction and $i,j = 1:6$. The added mass tensor, m_{ij}, is symmetric!

To find the 3D added mass coefficients consider simply the body geometry, ignoring for now the actual motions of the vessel. To start, orient the 1-axis along the long axis of the slender body as shown in figure 1. The 3D added mass coefficients will be found by summing (or integrating) the added mass coefficients of the 2D cross-sectional slices along the body.
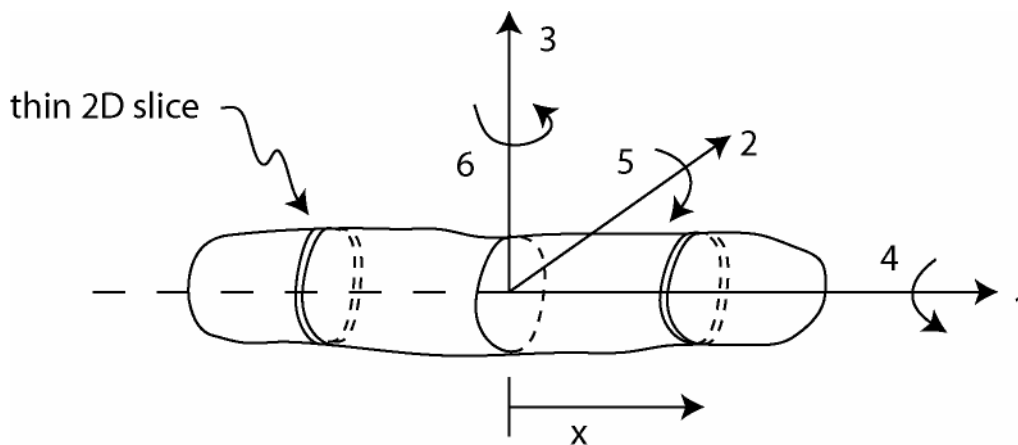

Figure 1: Slender body oriented with the long axis in the 1-direction.

The sectional added mass coefficients are tabulated for simple geometries. In general, with the slender body aligned lengthwise along the 1-axis, the 2D cross-sectional slice is aligned with the 2-3 plane, some distance *x* from the origin (figure 1). This 2D slice is shown in figure 2. To find the 3D coefficients we need to know the 2D coefficient of each section (strip) along the length of the vessel. For a uniform diameter cylinder this is quite simple, but for ships with complex geometry there is a bit more work involved.
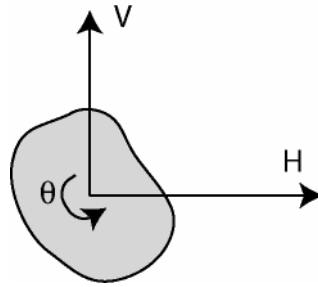

Figure 2: 2D cross-sectional slice of slender body.

The 2D coefficients will be written as a_{ij} whereas the 3D coefficients are written as m_{ij}.

1. $a_{HH} \rightarrow$ Horizontal force due to a unit horizontal acceleration
2. $a_{VV} \rightarrow$ Vertical force due to a unit vertical acceleration
3. $a_{\theta\theta} \rightarrow$ Moment about the origin due to a unit rotational acceleration
4. $a_{HV} \rightarrow$ Horizontal force due to a unit vertical acceleration
5. $a_{H\theta} \rightarrow$ Horizontal force due to unit rotational acceleration
6. $a_{V\theta} \rightarrow$ Vertical force due to unit rotational acceleration

For a slender body with the 1 axis aligned along the "long" axis, each of the 2D coordinates corresponds to a 3D coordinate as follows:

* *H-axis* $\leftrightarrow$ the +2 axis
* *V-axis* $\leftrightarrow$ the +3 axis
* *θ-axis* $\leftrightarrow$ the 4 rotation

To determine the total 3D added mass on a body we need to:

1. Determine the acceleration at each cross-sectional slice for a unit 3D body acceleration
2. Multiply this acceleration by the appropriate 2D added mass coefficient to get the

    force at that section in the ith direction.
3. Integrate these forces over the length of the body.

Added mass coefficients in 3D:

- $m_{33} \rightarrow$ the force in the 3 direction due to a unit 3 acceleration. A unit 3 acceleration in 3D is like a unit acceleration in the +V-direction for the 2D section ($\dot{U}_3 = \dot{V}_V$). Therefore, for each section the force per unit length is $F_3 = a_{VV}(x)\,\dot{V}_V$. For a unit acceleration $\dot{U}_3 = \dot{V}_V = 1$. Integrating the force per length F_3 over the body we get:

$$m_{33} = \int_L a_{VV}(x)\, dx$$

- $m_{22} \rightarrow$ the force in the 2 direction due to a unit 2 acceleration. A unit 2 acceleration in 3D is like a unit acceleration in the +H-direction for the 2D section ($\dot{U}_2 = \dot{V}_H$). Therefore, for each section the force per unit length is $F_2 = a_{HH}(x)\,\dot{V}_H$. For a unit acceleration $\dot{U}_2 = \dot{V}_H = 1$. Integrating the force per length F_2 over the body we get:

$$m_{22} = \int_L a_{HH}(x)\, dx$$

- $m_{32} \rightarrow$ the force in the 3 direction due to a unit 2 acceleration. A unit 2 acceleration in 3D is like a unit acceleration in the +H-direction for the 2D section ($\dot{U}_2 = \dot{V}_H$) as in m_{22}. Therefore, for each section the force per unit length in 3 is $F_3 = a_{HV}(x)\,\dot{V}_H$. For a unit acceleration $\dot{U}_2 = \dot{V}_H = 1$. Integrating the force per length F_3 over the body we get:

$$m_{32} = \int_L a_{HV}(x)\, dx$$

- $m_{44} \rightarrow$ moment in the 4-direction (roll) due to a unit rotational acceleration in 4. This acceleration corresponds to a θ rotation in 2D ($\dot{U}_4 = \dot{V}_\theta$). The moment (per length) in the θ direction corresponds to a moment in the 4 direction $F_4 = a_{\theta\theta}\,\dot{V}_\theta$. Integrating over the body length,

$$m_{44} = \int_L a_{\theta\theta}(x)\, dx$$

- $m_{24} \rightarrow$ Force in the 2-direction due to unit 4 acceleration. A rotation in 4 corresponds to a rotation in θ ($\dot{U}_4 = \dot{V}_\theta$). A force in 2 corresponds to a force in the horizontal direction, $F_2 = a_{H\theta}(x)\,\dot{V}_\theta$. For $\dot{U}_4 = \dot{V}_\theta = 1$, $F_2 = a_{H\theta}(x)$ so

$$m_{24} = \int_L a_{H\theta}(x)\,dx$$

- $m_{34} \rightarrow$ Force in the 3-direction due to unit 4 acceleration. A rotation in 4 corresponds to a rotation in θ ($\dot{U}_4 = \dot{V}_\theta$). A force in 3-direction (3D) corresponds to a force in the vertical direction (2D), $F_3 = a_{V\theta}(x)\,\dot{V}_\theta$. For $\dot{U}_4 = \dot{V}_\theta = 1$, $F_2 = a_{H\theta}(x)$ so

$$m_{34} = \int_L a_{V\theta}(x)\,dx$$

- $m_{35} \rightarrow$ Force in the 3-direction due to unit 5 (pitch) acceleration. The 2D sectional motion due to pitch (5 acceleration) is essentially a vertical acceleration or the acceleration in 5 (about the origin) times the moment arm x, the distance from the origin of the slice: $x\dot{U}_5 = \dot{V}_V$. The 2D force due to rotation is $F_3 = a_{VV}(x)\,\dot{V}_V = a_{VV}(x)\,x\dot{U}_5$. For a unit acceleration, integrating the force per length F_3 over the body we get:

$$m_{35} = \int_L x\,a_{VV}(x)\,dx$$

- $m_{25} \rightarrow$ Force in the 2-direction due to unit 5 (pitch) acceleration. The 2D sectional motion due to pitch (5 acceleration) is essentially a vertical acceleration or the acceleration in 5 (about the origin) times the moment arm x, the distance from the origin of the slice: $x\dot{U}_5 = \dot{V}_V$. The 2D force due to rotation is $F_2 = a_{VH}(x)\,\dot{V}_V = a_{VH}(x)\,x\dot{U}_5$. For a unit acceleration, integrating the force per length F_3 over the body we get:

$$m_{25} = \int_L x\,a_{VH}(x)\,dx$$

- $m_{45} \rightarrow$ Moment in the 4-direction due to unit 5 (pitch) acceleration. Again we have: $x\dot{U}_5 = \dot{V}_V$. The 2D moment due to rotation is $F_4 = F_\theta = a_{V\theta}(x)\,\dot{V}_V = a_{V\theta}(x)\,x\dot{U}_5$. For a unit acceleration, integrating the moment per length F_4 over the body we get:

$$m_{45} = \int_L x\,a_{V\theta}(x)\,dx$$

- $m_{55} \rightarrow$ Moment in the 5-direction due to unit 5 (pitch) acceleration. Again we have: $x\dot{U}_5 = \dot{V}_V$. The 2D moment due to rotation is $F_5 = x\,F_V = x\,a_{VV}(x)\,\dot{V}_V = x^2\,a_{VV}(x)\,\dot{U}_5$. For a unit acceleration, integrating the moment per length F_5 over the body we get:

$$m_{55} = \int_L x^2\,a_{VV}(x)\,dx$$

- $m_{26} \rightarrow$ Force in the 2-direction due to unit 6 (yaw) acceleration. The 2D sectional motion due to yaw (6 acceleration) is essentially a horizontal acceleration or the acceleration in 6 (about the origin) times the moment arm x, the distance from the origin of the slice: $x\dot{U}_6 = \dot{V}_H$. The 2D force due to rotation is $F_2 = a_{HH}(x)\,\dot{V}_H = a_{VV}(x)\,x\dot{U}_6$. For a unit acceleration, integrating the force per length F_3 over the body we get:

$$m_{26} = \int_L x\,a_{HH}(x)\,dx$$

- $m_{36} \rightarrow$ Force in the 3-direction due to unit 6 (yaw) acceleration. The 2D sectional motion due to yaw (6 acceleration) is essentially a horizontal acceleration or the acceleration in 6 (about the origin) times the moment arm x, the distance from the origin of the slice: $x\dot{U}_6 = \dot{V}_H$. The 2D force due to rotation is $F_3 = a_{VH}(x)\,\dot{V}_H = a_{VH}(x)\,x\dot{U}_6$. For a unit acceleration, integrating the force per length F_3 over the body we get:

$$m_{36} = \int_L x\,a_{VH}(x)\,dx$$

- $m_{46} \rightarrow$ Moment in the 4-direction due to unit 6 (yaw) acceleration. Again: $x\dot{U}_6 = \dot{V}_H$. The 2D moment due to rotation is $F_4 = a_{H\theta}(x)\,\dot{V}_H = a_{V\theta}(x)\,x\dot{U}_6$. For a unit acceleration, integrating we get:

$$m_{46} = \int_L x\,a_{H\theta}(x)\,dx$$

- $m_{56} \rightarrow$ Moment in the 5-direction due to unit 6 (yaw) acceleration. Again: $x\dot{U}_6 = \dot{V}_H$. The 2D moment due to rotation in 5 is due to the 2D force in the vertical direction acting with a lever arm x: $F_5 = x\,F_V = x\,a_{HV}(x)\,\dot{V}_V = x^2\,a_{HV}(x)\,\dot{U}_6$. For a unit acceleration, integrating we get:

$$m_{56} = \int_L x^2\,a_{HV}(x)\,dx$$

- $m_{66} \rightarrow$ Moment in the 6-direction due to unit 6 (yaw) acceleration. Again: $x\dot{U}_6 = \dot{V}_H$.

  The 2D moment due to rotation is $F_6 = x F_H = x a_{HH}(x) \dot{V}_H = x^2 a_{HH}(x) \dot{U}_6$. For a unit acceleration, integrating we get:

$$m_{66} = \int_L x^2 \, a_{HH}(x) \, dx$$

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	1	2	3	4	5	6
1	m_{11}	m_{12}	m_{13}	m_{14}	m_{15}	m_{16}
2		$m_{22} = \int_L a_{HH} \, dx$	$m_{23} = \int_L a_{HV}(x) \, dx$	$m_{24} = \int_L a_{H\theta}(x) \, dx$	$m_{25} = \int_L x \, a_{VH}(x) \, dx$	$m_{26} = \int_L x \, a_{HH} \, dx$
3			$m_{33} = \int_L a_{VV} \, dx$	$m_{34} = \int_L a_{V\theta}(x) \, dx$	$m_{35} = \int_L x \, a_{VV} \, dx$	$m_{36} = \int_L x \, a_{VH}(x) \, dx$
4				$m_{44} = \int_L a_{\theta\theta} \, dx$	$m_{45} = \int_L x \, a_{V\theta}(x) \, dx$	$m_{46} = \int_L x \, a_{H\theta}(x) \, dx$
5					$m_{55} = \int_L x^2 \, a_{VV} \, dx$	$m_{56} = \int_L x^2 \, a_{HV}(x) \, dx$
6						$m_{66} = \int_L x^2 \, a_{HH}(x) \, dx$

Empty Boxes can be filled in my symmetry. $m_{ij} = m_{ji}$

More reading can be found in the handbook: <u>Principles of Naval Architecture Vol III.</u>, (1989) Soc. Naval Arch. and Marine Engineers, p. 56.