13.012 HW#3
Solutions 2004

Problem 1

1. Boundary Conditions

   Free surface: Pressure across interface is constant: $P = P_{atm}$, A particle on the surface stays on the surface.

   $$\boxed{\begin{array}{ll} \frac{\partial^2\phi}{\partial t^2} + g\,\frac{\partial\phi}{\partial z} = 0 & \\ \eta = -\frac{1}{g}\,\frac{\partial\phi}{\partial t} & \text{on } z = 0. \end{array}}$$

   Bottom: No vertical velocity at the bottom ($w = 0$).

2. Particle Motion

   Particles move in elliptical orbits underneath surface waves. For deep water waves the paths are circular.

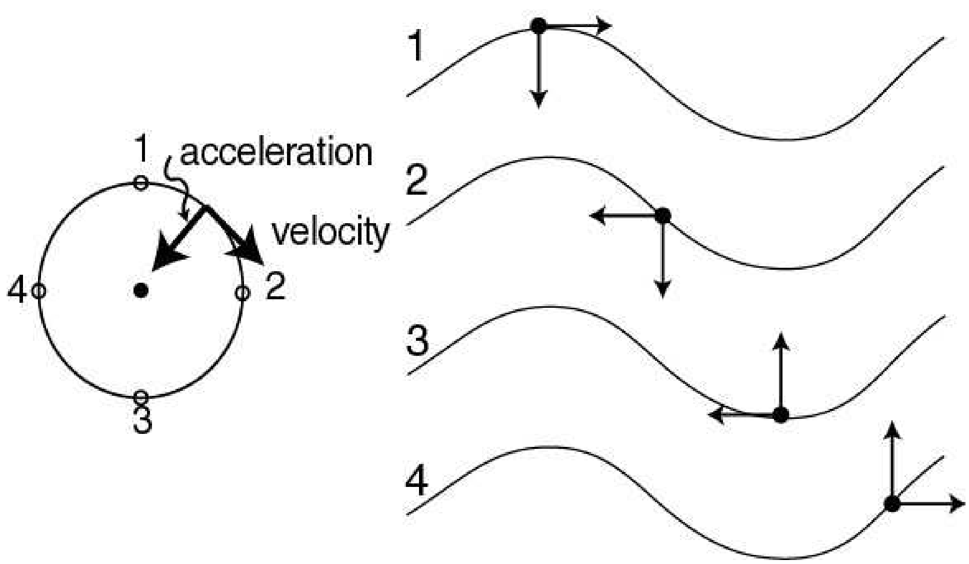

Figure 1: Velocity and Acceleration for particles at four points in cycle. 1-4: Crest, nodal point, trough, nodal point. Acceleration is inwards and velocity is tangential.

At position # 1 the velocity is to the right and acceleration is downwards, at 2 the velocity is down and acceleration is to the left, at 3 the velocity is left and the acceleration is upwards, and at position 4 the velocity is upwards and acceleration to the right.

Looking at the plots from part II: the maximum magnitude of the x-velocity occurs at the trough and crests, the max. magnitude of horizontal acceleration at the nodal points, the max. vertical velocity at the nodal points and the max. vertical acceleration at the trough and crest (note that the vertical acceleration is 180° out of phase from the surface elevation, η.)

3. Group and Phase Velocity

   Deep Water: $H \to \infty$

   $$\boxed{\omega^2 = kg \text{ and } C_g = \frac{1}{2}C_p}$$

   Shallow Water: $H \to 0$

   $$\boxed{\omega = \sqrt{gH}k \text{ and } C_g = C_p}$$

   Group Speed is the speed at which a packet (or group) of waves travels it is also the speed at which the energy moves.

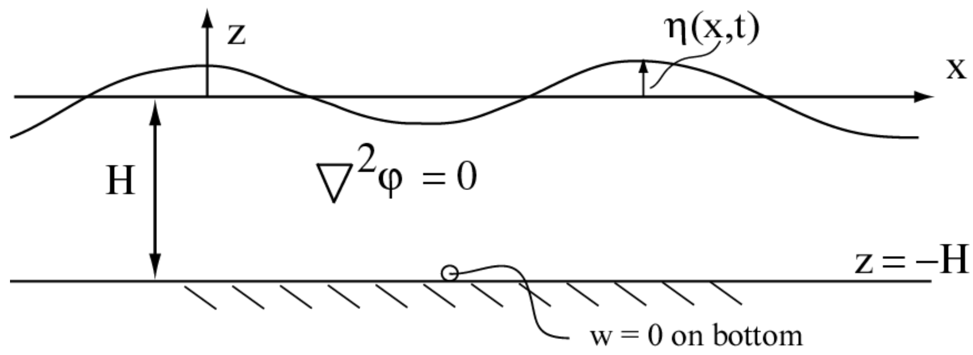

Figure 2: Schematic for Plane progressive Waves

Problem 2

Given a linear wave propagating in the direction of positive x defined as in figure 2, we write the potential as

$$\phi(x,z,t) = \frac{A\omega}{k}\frac{\cosh\left(k\left(z+H\right)\right)}{\sinh\left(kH\right)}\sin\left(kx-\omega t\right). \tag{1}$$

and the surface elevation, η, as

$$\eta(x,t) = A\cos(kx-\omega t). \tag{2}$$

In deep water the potential (1) becomes

$$\phi(x,z,t) = \frac{A\omega}{k}e^{kz}\sin\left(kx-\omega t\right). \tag{3}$$

From the potential (3) we can write dynamic pressures, velocities and accelerations for a deep water wave as

$$\begin{aligned}
p(x,z,t) &= -\rho\frac{\partial\phi}{\partial t} = \rho\frac{a\omega^2}{k}e^{kz}\cos\left(kx-\omega t\right), && (4)\\
u(x,z,t) &= \frac{\partial\phi}{\partial x} = a\omega e^{kz}\cos\left(kx-\omega t\right), && (5)\\
w(x,z,t) &= \frac{\partial\phi}{\partial z} = a\omega e^{kz}\sin\left(kx-\omega t\right), && (6)\\
u'(x,z,t) &= \frac{\partial u}{\partial t} = a\omega^2 e^{kz}\sin\left(kx-\omega t\right), && (7)\\
w'(x,z,t) &= \frac{\partial w}{\partial t} = -a\omega^2 e^{kz}\cos\left(kx-\omega t\right). && (8)
\end{aligned}$$

Be careful of the signs! Recall that

$$\begin{aligned}
\sin\left(-x\right) &= -\sin\left(x\right), && (9)\\
\cos\left(-x\right) &= \cos\left(x\right). && (10)
\end{aligned}$$

Calculating the necessary constants

$$\begin{aligned}
k &= \frac{2\pi}{\lambda} = 0.0483 \text{ ft}^{-1},\\
\omega &= \sqrt{gk} = 0.695 \text{ rad s}^{-1},\\
z_n &= \frac{m\lambda}{110} = 15.36 \text{ ft},.
\end{aligned}$$

and evaluating equations 4 through 8 with $z = z_n$ (m = 13) and t fixed at zero and plotting the results as a function of non-dimensional x yields the results shown in figure 3.

If we again fix $z = z_n$, but now hold x constant while we plot evaluations of equations 4 through 8 as functions of non-dimensional time (period $T = 2\pi/\omega$) we get the results shown in figure 4.

1. The inviscid force is an added mass effect and thus is proportional to fluid acceleration. Looking at the results for u' in figure 3 we see that the horizontal acceleration has maximum amplitude under the points where $\eta = 0$ (wave nodal points). For the vertical force, we see that w' has maximum amplitude at wave crests and troughs.

2. With viscous forces proportional to the square of the fluid velocity, maximum forces will occur at points of maximum velocity. From the plot of u in figure 3 we see that maximum horizontal viscous forces will occur under wave crests and troughs. For the vertical force, we see that w has maximum amplitude under wave nodal points.

3. The total pressure at a point is the sum of the ambient, hydrostatic, and dynamic pressures

$$p = p_{atm} + p_h + p_d. \tag{11}$$

   The pressure that we measure at depth H_1 under the still water surface is the result from the first of these two terms. From figure 3 and equation 4 we can quickly obtain results for the dynamic pressure under the crests, troughs and nodal points,

$$\begin{aligned} p_{crest} &= \rho a g e^{-kH_1} + \rho g H_1 + p_{atm}, & (12)\\ p_{trough} &= -\rho a g e^{-kH_1} + \rho g H_1 + p_{atm}, & (13)\\ p_{node} &= \rho g H_1 + p_{atm}. & (14) \end{aligned}$$

   The z-coordinate is referenced from the mean free surface so we measure hydrostatic pressure at a distance H_1 below the still water line. We employ the free surface condition on $z = 0$ (as opposed to $z = \eta$) because this is linear theory ($A/\lambda << 1$) and when we expand ϕ in a Taylor series in the neighborhood of the free surface

$$\phi(x, z = \eta, t) = \phi(x, 0, t) + \eta \frac{\partial \phi}{\partial z} + \text{HOT}, \tag{15}$$

   we drop all second- and higher-order terms and are left with the potential evaluated at $z = 0$.

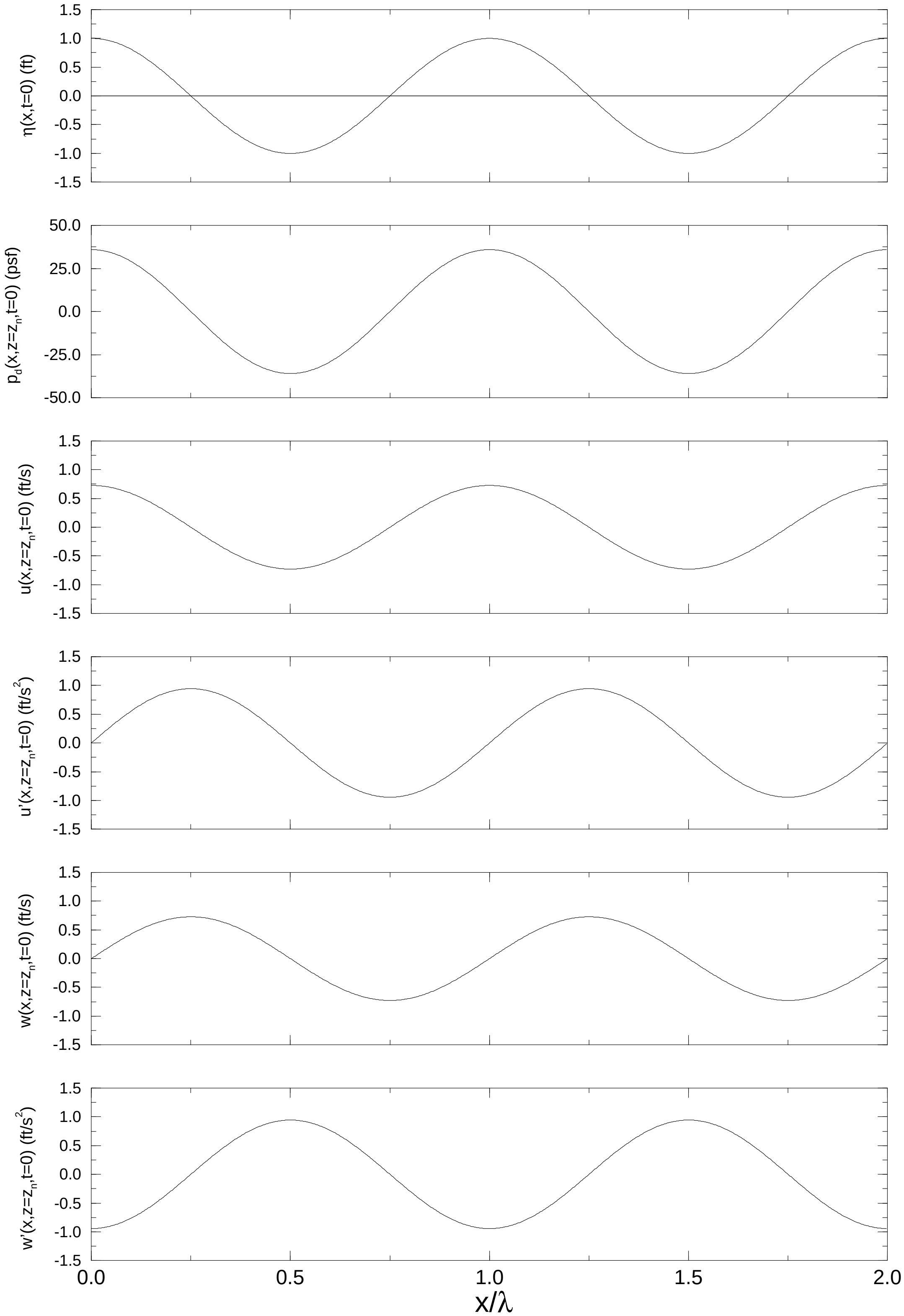

Figure 3: Features of a linear wave at a fixed time: $t = 0$.

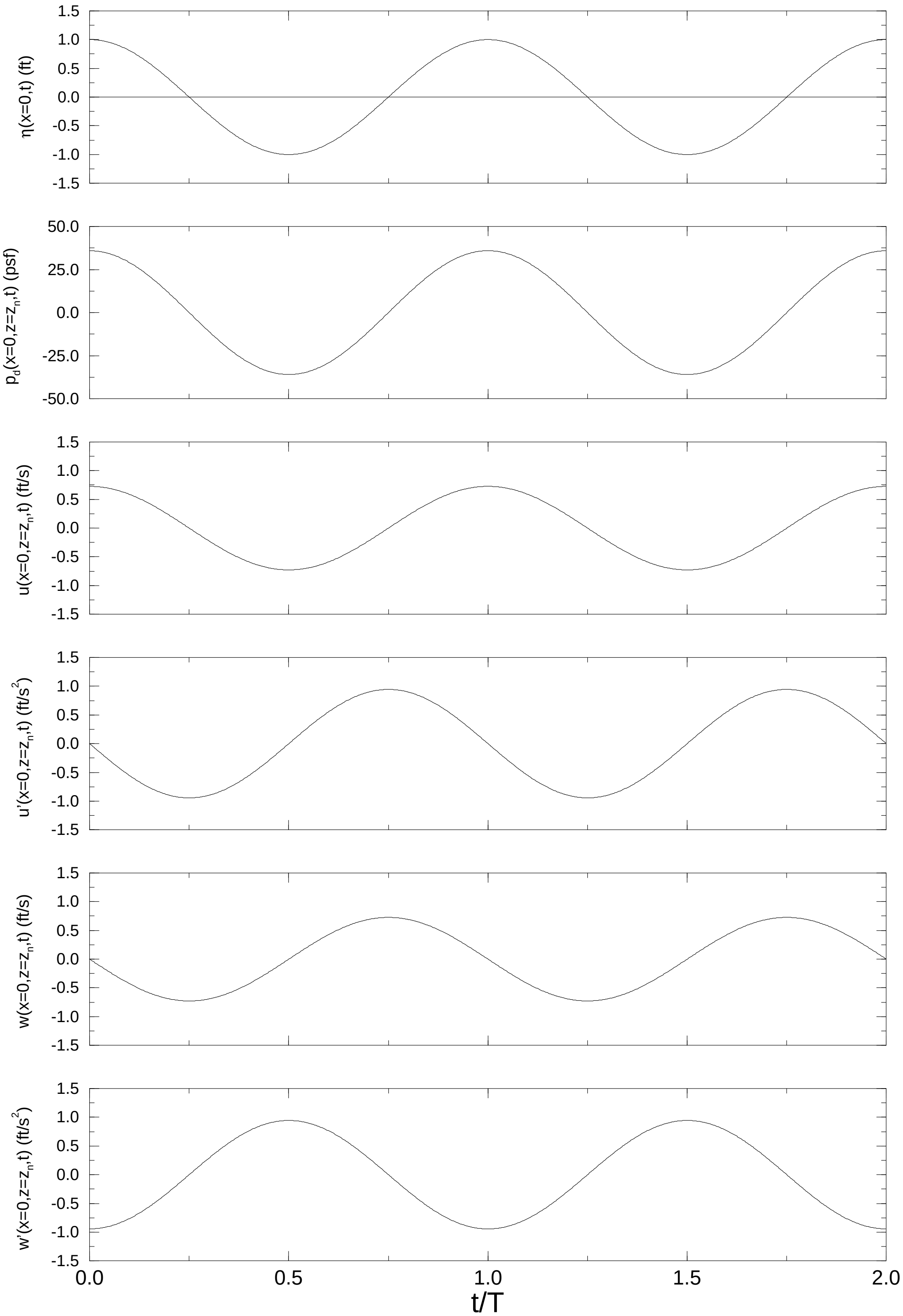

Figure 4: Features of a linear wave at a fixed point: $x = 0$.