Introduction

Governing Equations so far:

<table>
<thead>
<tr>
<th>Knowns</th>
<th>Number of Equations</th>
<th>Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ</td>
<td>Continuity (conservation of mass)</td>
<td>1</td>
</tr>
<tr>
<td>( F_i )</td>
<td>Euler (conservation of momentum)</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ \text{4 of 9 eliminated by symmetry} \]

Therefore, some constitutive relationships are needed to relate \( v_i \) to \( \tau_{ij} \).

1.7 Newtonian Fluid

(i) Consider a fluid at rest \((v_i \equiv 0)\). Then according to Pascal’s Law, the shear stresses on a small area \( \delta_{ij} \) is:

\[
\tau_{ij} = -p_s \delta_{ij} \quad \text{(Pascal’s law)}
\]

Hydrostatic pressure

(ii) Consider a fluid in motion. The fluid stress is defined as:

\[
\tau_{ij} = -p \delta_{ij} + \hat{\tau}_{ij}
\]

Thermodynamic pressure Dynamic stress

The dynamic viscous stress \( \tau_{ij} \) can be related to velocities empirically.

Experiments with a wide class of fluids obtain that, for most fluids, \( \tau_{ij} \) is a linear function of rate of strain and velocity gradient. Fluids with these characteristics are ‘Newtonian’ fluids.

Rate of strain:

\[
\frac{\partial}{\partial t} \left( \frac{\partial X}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial X}{\partial u} \right)
\]

Note that the shear stress is proportional to the rate of strain.
Velocity gradient: $\frac{\partial u_k}{\partial x_m}$

i.e. $\hat{\tau}_{ij} \approx \alpha_{ijklm} \frac{\partial u_k}{\partial x_m}$ \quad i, j, k, m = 1, 2, 3

$3^5 = 81$ empirical coefficients (constants for Newtonian fluid)

For isotropic fluids, this reduces to:

$$\hat{\tau}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_i}{\partial x_i} \delta_{ij}$$

Where the fluid properties are $\mu$: coefficient of dynamic viscosity and $\lambda$: bulk elasticity, ‘second’ coefficient of viscosity

For incompressible flow, $\frac{\partial u_i}{\partial x_i} = 0$

Therefore, for an incompressible, isotropic, Newtonian fluid:

$$\hat{\tau}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$\hat{\tau}_{ij}$ is the viscous stress

1.8 Navier-Stokes equations

<table>
<thead>
<tr>
<th>Euler and continuity</th>
<th># of equations</th>
<th># of unknowns</th>
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<tr>
<td>1</td>
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<tr>
<td>6 (symmetry)</td>
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<td>10</td>
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Closure

Substitute the Newtonian equation:

$$\tau_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

into Euler’s equation:

$$\rho \frac{Du_i}{Dt} = F_i + \frac{\partial \tau_{ij}}{\partial x_j}$$
\[
\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

where

\[
\frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_i}
\]

due to continuity

Finally,

\[
\begin{align*}
\frac{D u_i}{D t} & = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho} F_i \\
\frac{D \vec{v}}{D t} & = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{F}
\end{align*}
\]

Tensor form

\[
\begin{align*}
\frac{D u_i}{D t} & = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho} F_i \\
\frac{D \vec{v}}{D t} & = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{F}
\end{align*}
\]

Vector form

where \( \nu \equiv \frac{\mu}{\rho} \) Kinematic viscosity \([L^2/T]\)

Navier-Stokes equations for incompressible, Newtonian fluids

<table>
<thead>
<tr>
<th>continuity</th>
<th># of equations</th>
<th>Navier-Stokes</th>
<th># of unknowns</th>
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<td></td>
<td>3</td>
<td>p</td>
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1.9 Boundary Conditions

(i) Kinematic Boundary Conditions

Specifies kinematics (position, velocity, ...)

On a solid boundary, velocity of the fluid = velocity of the body. i.e. velocity continuity

\[
\vec{v} = \vec{u}_{\text{body}} \quad \text{"no-slip" boundary condition}
\]

- \( \vec{v} \cdot \vec{n} = \vec{u} \cdot \vec{n} \) no flux --- continuous flow
- \( \vec{v} \cdot \vec{t} = \vec{u} \cdot \vec{t} \) no slip --- finite shear stress
(ii) Dynamic Boundary Conditions

Specifies dynamics (pressure, shear stress, …)

Stress continuity:

\[ p' + \tau_{ij} = p + \tau_{ij} \]

\[ p = p' + p_{\text{interface}} \]

\[ \tau_{ij} = \tau_{ij}' + \tau_{ij\text{interface}} \]

The most common example of interfacial stress is surface tension.

**Surface Tension**

The presence of surface tension implies that there exists a pressure jump across the interface. Therefore, \( \Sigma \) gives a \( \Delta p \) across an interface.

Surface Tension: \( \Sigma \) [Tension force / Length] \( \equiv \) [Surface energy / Area]

For a water/air interface: \( \Sigma = 0.07 \text{ N/m} \) This is a function of temperature, impurities etc…

2D Example:

\[
\cos \frac{d\theta}{2} \cdot \Delta p \cdot R d\theta = 2\Sigma \sin \frac{d\theta}{2} = 2\Sigma \frac{d\theta}{2}
\]

\[
\therefore \Delta p = \frac{\Sigma}{R}
\]

Concave has higher pressure.

3D Example: Compound curvature

\[
\Delta p = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \Sigma
\]
where \( R_1 \) and \( R_2 \) are the principle radii of curvature.

### 1.10 Body Forces – Gravity

- Conservative force:

\[
\mathbf{F} = -\nabla \phi \quad \text{for some } \phi \quad \text{where } \phi \text{ is the force potential}
\]

\[
\oint \mathbf{F} \cdot d\mathbf{x} = 0
\]

A special case of a conservative force is gravity.

\[
\mathbf{F} = -\rho g \mathbf{k}
\]

Gravitational potential: \( \phi = \rho gz \)

\[
\mathbf{F} = -\nabla \phi = \nabla (-\rho gz) = -\rho g \mathbf{k}
\]

Navier-Stokes:

\[
\rho \frac{D \mathbf{V}}{Dt} = -\nabla p + \mathbf{F} + \rho \nabla \mathbf{V} \cdot \mathbf{V}
\]

\[
= -\nabla p - \rho g z + \rho \mathbf{V} \cdot \nabla \mathbf{V}
\]

\[
= -\nabla (p + \rho g z) + \rho \mathbf{V} \cdot \nabla \mathbf{V}
\]

\( \downarrow \quad \text{total pressure} \quad \text{dynamic pressure} \)

\( \downarrow \quad \text{but } p - p_s = p_d \quad \text{and } p_s = \rho g z \)

Therefore

\[
\rho \frac{D \mathbf{V}}{Dt} = -\nabla p_d + \rho \mathbf{V} \cdot \nabla \mathbf{V}
\]

Presence of gravity body force is equivalent to replacing total pressure \( p \) by dynamic pressure \( p_d \) in the Navier-Stokes(N-S) equation.

Solve the N-S equation with \( p_d \), then calculate \( p = p_d + p_s = p_d - \rho g z \)