13.021 - Marine Hydrodynamics
Lecture 10

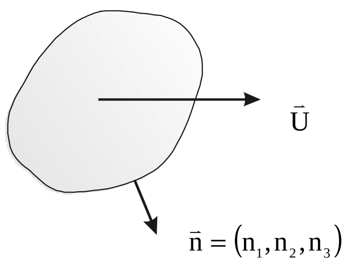
3.5 - Boundary Conditions for Potential Flow

Types of Boundary Conditions:

1. Kinematic Boundary Conditions - specify the flow velocity $\vec{v}$ at boundaries.
2. Dynamic Boundary Conditions - specify force $\vec{F}$ or pressure p at flow boundary.

- Kinematic Boundary Conditions on an impermeable boundary (no flux condition):

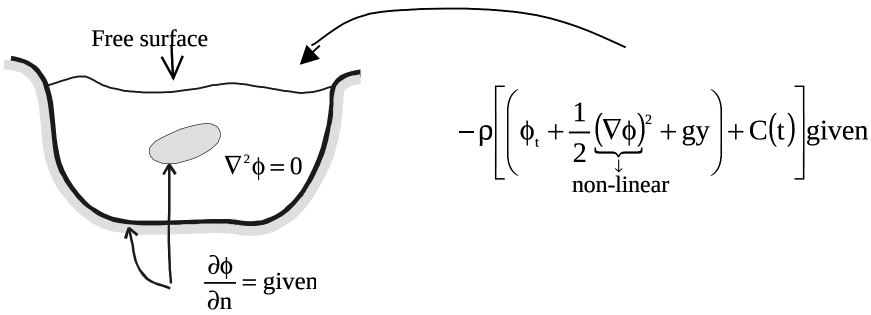
$$\begin{aligned}&\vec{n}\cdot\vec{v} = \vec{n}\cdot\vec{U} = U_n \;\; \text{(given)} \;\; \text{where} \;\; \vec{v} = \nabla\phi \\ &\vec{n}\cdot\nabla\phi = U_n \longrightarrow \frac{\partial\phi}{\partial n} = U_n \\ &\frac{\partial}{\partial n} = \left(n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2} + n_3\frac{\partial}{\partial x_3}\right)\end{aligned}$$



- Dynamic Boundary Conditions: Pressure is prescribed in general.

$$p = -\rho\left(\phi_t + \frac{1}{2}(\nabla\phi)^2 + gy\right) + C(t) \;\; \text{(prescribed)}$$

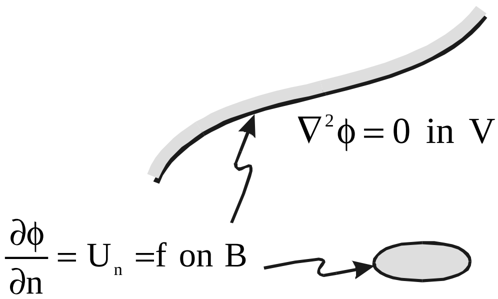
In general:



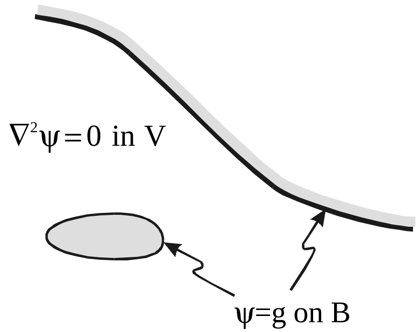
Linear Superposition for Potential Flow

In the **absence** of **dynamic boundary conditions**, the potential flow boundary value problem is **linear**.

- Potential function ϕ.



- Stream function ψ.



Linear Superposition: if $\phi_1, \phi_2, \ldots$ are harmonic functions, i.e. $\nabla^2 \phi_i = 0$, then $\phi = \sum \alpha_i \phi_i$, where α_i are constants, are also harmonic, and is the solution for the boundary value problem provided the boundary conditions (kinematic boundary condition) are satisfied, i.e.

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}(\alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots) = U_n \quad \text{on} \quad \text{B}.$$

The key is to combine known solution of the Laplace equation in such a way as to satisfy the K.B.C. (kinematic boundary condition).
The same is true for the stream function ψ. K.B.C.s specify the value of ψ on the boundaries.

Example:

$\phi_i\left(\vec{x}\right)$is a unit-source flow with source at $\vec{x}_i$
i.e.

$$\begin{aligned} \phi_i\left(\vec{x}\right) \equiv \phi_{\text{source}}\left(\vec{x}, \vec{x}_i\right) &= \frac{1}{2\pi} \ln \left|\vec{x} - \vec{x}_i\right| \qquad \text{(in 2D)} \\ &= -\left(4\pi \left|\vec{x} - \vec{x}_i\right|\right)^{-1} \text{(in 3D)}, \end{aligned}$$

then find m_i such that:

$$\phi = \sum_i m_i \phi_i(\vec{x}) \;\; \text{satisfies KBC on B}$$

Caution: ϕ must be regular for $x \in V$, so it is required that $\vec{x} \notin V$.

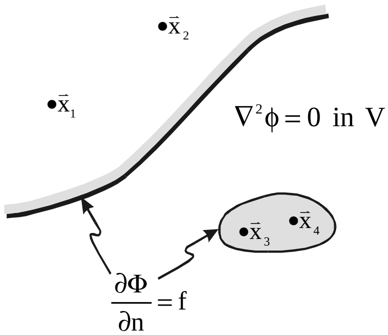


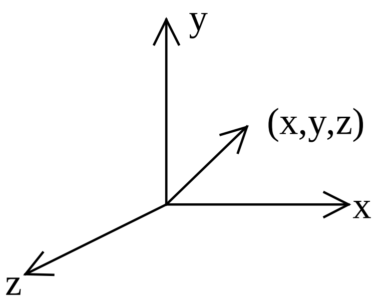
Figure 1: Note: $\vec{x}_j, j = 1, \ldots, 4$ are *not* in the fluid domain V.

Laplace equation in different coordinate systems (cf Hildebrand §6.18)

1. **Cartesian (x,y,z)**

$$\vec{v} = \left(\overset{\hat{i}}{u}, \overset{\hat{j}}{v}, \overset{\hat{k}}{w}\right) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

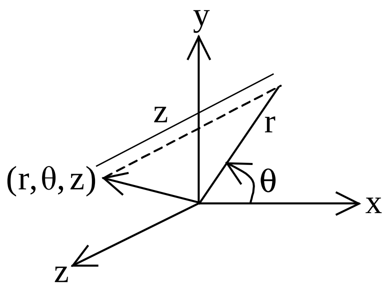


2. **Cylindrical (r,θ,z)**

$$r^2 = x^2 + y^2,$$
$$\theta = \tan^{-1}(y/x)$$

$$\vec{v} = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_z}{v_z}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \left(\underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial\phi}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

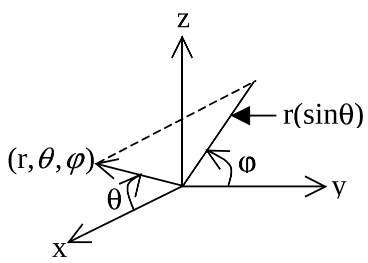


3. **Spherical (r,θ,φ)**

$$r^2 = x^2 + y^2 + z^2,$$
$$\theta = \cos^{-1}(x/r) \text{ or } x = r\,(\cos\theta)$$
$$\varphi = \tan^{-1}(z/y)$$

$$\vec{v} = \nabla\phi = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_\varphi}{v_\varphi}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r(\sin\theta)}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\nabla^2\phi = \left(\underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} \right)$$



3.7 - Simple Potential flows

1. **Uniform Stream:** $\nabla^2(ax+by+cz+d)=0$

1D:	$\phi = Ux +$ constant	$\psi = Uy +$ constant	$\vec{v} = (U,0,0)$
2D:	$\phi = Ux + Vy +$ constant	$\psi = Uy - Vx +$ constant	$\vec{v} = (U,V,0)$
3D:	$\phi = Ux + Vy + Wz +$ constant		$\vec{v} = (U,V,W)$

2. **Source (sink) flow:**

2D: Polar coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \text{ with } r = \sqrt{x^2+y^2}$$

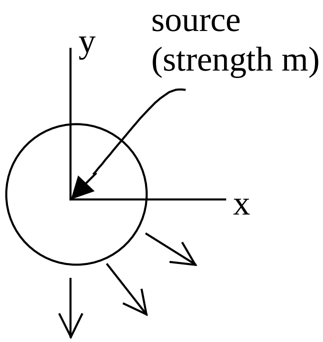
An axisymmetric solution: $\phi = \ln r$ (verify)

Define 2D source of strength m at r = 0:

$$\phi = \frac{m}{2\pi} \ln r$$

It satisfies $\nabla^2 \phi = 0$, except at $r = \sqrt{x^2 + y^2} = 0$ (so must exclude r = 0 from flow)

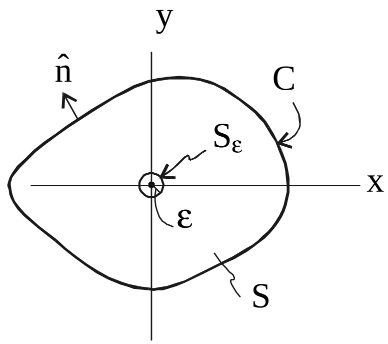
$$\nabla \phi = \frac{\partial \phi}{\partial r} \hat{e}_r = \frac{m}{2\pi r} \hat{e}_r, \text{ i.e. } \quad v_r = \frac{m}{2\pi r}, v_\theta = 0$$



Net outward volume flux is

$$\oint_C \vec{v} \cdot \hat{n} ds = \iint_S \nabla \cdot \vec{v} ds = \iint_{S_\varepsilon} \nabla \cdot \vec{v} ds$$

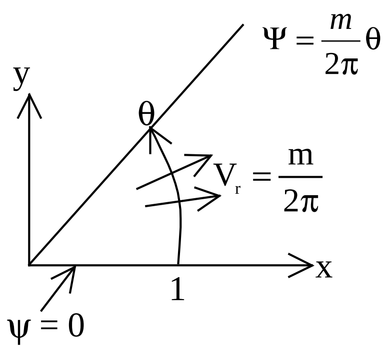
$$\oint_{C_\varepsilon} \vec{v} \cdot \hat{n} ds = \int_0^{2\pi} \underbrace{V_r}_{\frac{m}{2\pi r_\varepsilon}} r_\varepsilon d\theta = \underbrace{m}_{\text{source strength}}$$



If $m < 0 \Rightarrow$ sink. Source m at (x_0, y_0):

$$\phi = \frac{m}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\phi = \frac{m}{2\pi} \ln r \text{ (Potential function)} \quad \longleftrightarrow \psi = \frac{m}{2\pi}\theta \quad \text{(Stream function)}$$



3D: Spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}, \cdots\right), \text{ where } \; r = \sqrt{x^2 + y^2 + z^2}$$

A spherically symmetric solution: $\phi = \frac{1}{r}$ (verify $\nabla^2\phi = 0$ except at r = 0)

Define 3D source of strength m at r = 0:

$$\phi = -\frac{m}{4\pi r}, \text{ then } V_r = \frac{\partial \phi}{\partial r} = \frac{m}{4\pi r^2}, V_\theta, V_\varphi = 0$$

Net outward volume flux is

$$\oiint V_r dS = 4\pi r_\varepsilon^2 \cdot \frac{m}{4\pi r_\varepsilon^2} = m \ (m < 0 \text{ for a sink })$$

3. **2D point vortex**

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Another particular solution: $\phi = a\theta + b$ (verify $\nabla^2\phi = 0$ except at $r = 0$)

Define the potential for a point vortex of circulation Γ at $r = 0$:

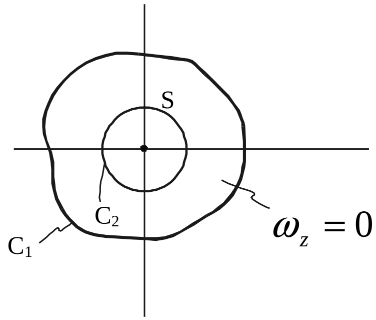
$$\phi = \frac{\Gamma}{2\pi}\theta, \text{ then } V_r = \frac{\partial \phi}{\partial r} = 0, V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \text{ and } \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rV_\theta) = 0 \text{ except at } r = 0$$

Stream function:

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

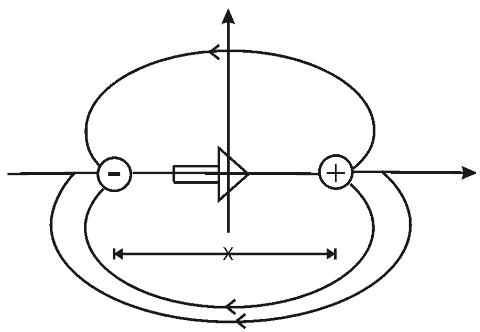
Circulation:

$$\int_{C_1} \vec{v} \cdot d\vec{x} = \int_{C_2} \vec{v} \cdot d\vec{x} + \underbrace{\int_{C_1 - C_2} \vec{v} \cdot d\vec{x}}_{\int\int_S \omega_z dS = 0} = \int_0^{2\pi} \frac{\Gamma}{2\pi r} r d\theta = \underbrace{\Gamma}_{\text{vortex strength}}$$



4. **Dipole (doublet flow)**

A **Dipole** is a **superposition** of a **sink** and a **source** with the same strength.



2D dipole:

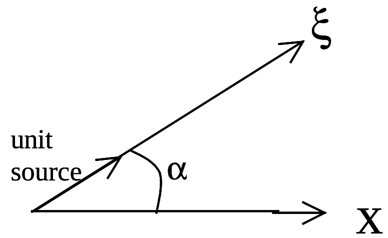
$$\phi = \frac{m}{2\pi}\left[\ln\sqrt{(x-a)^2+y^2} - \ln\sqrt{(x+a)^2+y^2}\right]$$

$$\lim_{a\to 0}\phi = \underbrace{\frac{\mu}{2\pi}}_{\substack{\mu = 2ma \\ \text{constant}}} \frac{\partial}{\partial\xi}\ln\sqrt{(x-\xi)^2+y^2}\bigg|_{\xi=0}$$

$$= -\frac{\mu}{2\pi}\frac{x}{x^2+y^2} = -\frac{\mu}{2\pi}\frac{x}{r^2}$$

2D dipole (doublet) of moment μ at the origin oriented in the +x direction.

NOTE: dipole = $\mu\frac{\partial}{\partial\xi}$ (unit source)



$$\phi = \frac{-\mu}{2\pi}\frac{x\cos\alpha + y\sin\alpha}{x^2+y^2} = \frac{-\mu}{2\pi}\frac{\cos\theta\cos\alpha+\sin\theta\sin\alpha}{r}$$

3D dipole:

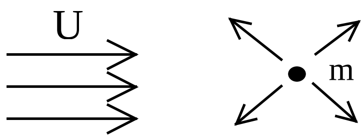
$$\phi = \lim_{a\to 0} \; -\frac{m}{4\pi}\left(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}\right) \text{ where } \mu = 2ma \text{ fixed}$$

$$= -\frac{\mu}{4\pi}\frac{\partial}{\partial\xi}\frac{1}{\sqrt{(x-\xi)^2+y^2+z^2}}\bigg|_{\xi=0} = -\frac{\mu}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mu}{4\pi}\frac{x}{r^3}$$

3D dipole (doublet) of moment μ at the origin oriented in the +x direction.

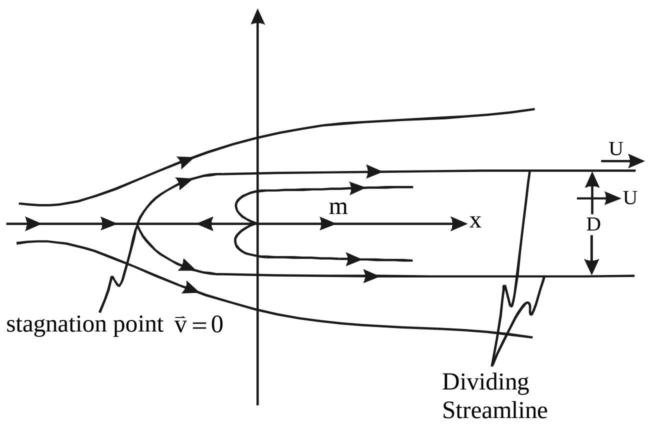
5. Stream and source: Rankine half-body

It is the **superposition** of a **uniform stream** of constant speed U and a **source** of strength m.



2D:

$$\phi = Ux + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}$$

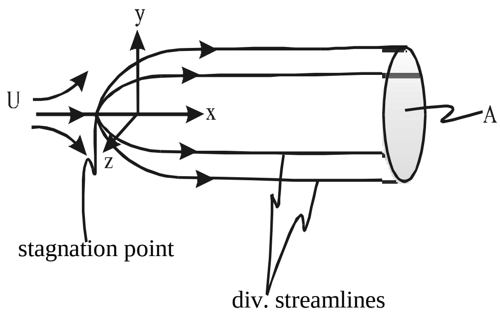


$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{2\pi} \frac{x}{x^2+y^2}$$
$$u\big|_{y=0} = U + \frac{m}{2\pi x}$$
$$\therefore u = 0 \text{ at } \quad x = x_s = -\frac{m}{2\pi U}$$

For large x, $u \to U$, and $UD = m$ by continuity, $\therefore D = \frac{m}{U}$.

3D:

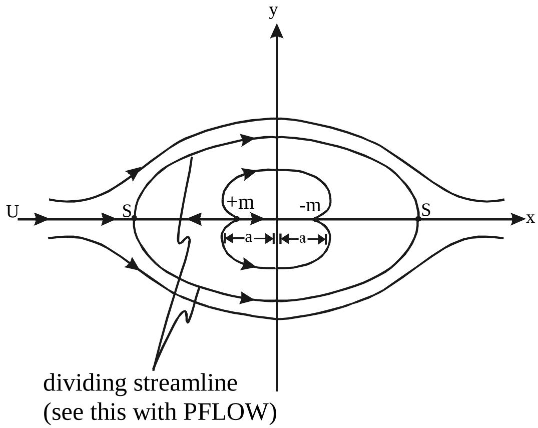
$$\phi = Ux - \frac{m}{4\pi\sqrt{x^2+y^2+z^2}}$$



$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}}$$
$$u\big|_{y=z=0} = U + \frac{m}{4\pi}\frac{x}{|x|^3}$$
$$\underset{v=w=0}{u=0} \text{ at } x = x_s = -\sqrt{\frac{m}{4\pi U}}$$

For large x, $u \to U$ and $UA = m$ by continuity, $\therefore A = \frac{m}{U}$.

6. **Stream + source/sink pair: Rankine closed bodies**



To have a closed body, a necessary condition is to have $\sum m_{\text{in body}} = 0$

2D Rankine ovoid:

$$\phi = Ux + \frac{m}{2\pi}\left(\ell n\sqrt{(x+a)^2 + y^2} - \ell n\sqrt{(x-a)^2 + y^2}\right)$$

3D Rankine ovoid:

$$\phi = Ux - \frac{m}{4\pi}\left[\frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}}\right]$$

For Rankine Ovoid,

$$\begin{aligned} u =& \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\left[\frac{x+a}{\left((x+a)^2 + y^2 + z^2\right)^{3/2}} - \frac{x-a}{\left((x-a)^2 + y^2 + z^2\right)^{3/2}}\right] \\ u|_{y=z=0} =& U + \frac{m}{4\pi}\left[\frac{1}{(x+a)^2} - \frac{1}{(x-a)^2}\right] \\ =& U + \frac{m}{4\pi}\frac{(-4ax)}{(x^2 - a^2)^2} \\ u|_{y=z=0} =& 0 \text{ at } \left(x^2 - a^2\right)^2 = \left(\frac{m}{4\pi U}\right) 4ax \end{aligned}$$

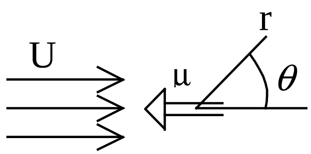
At $x = 0$,

$$u = U + \frac{m}{4\pi}\frac{2a}{(a^2+R^2)^{3/2}} \quad \text{where} \quad R = y^2 + z^2$$

Determine radius of body R_0:

$$2\pi\int_0^{R_0} uRdR = m$$

7. **Stream + Dipole: circles and spheres**

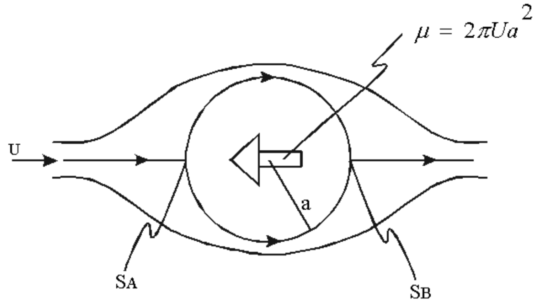


2D:

$$\begin{aligned}\phi =& Ux + \frac{\mu x}{2\pi r^2}, \quad \text{where} \quad x = r\cos\theta \\ =& \cos\theta\left(Ur + \frac{\mu}{2\pi r}\right) \quad \text{then} \quad V_r = \frac{\partial\phi}{\partial r} = \cos\theta\left(U - \frac{\mu}{2\pi r^2}\right)\end{aligned}$$

So $V_r = 0$ on $r = a = \sqrt{\frac{\mu}{2\pi U}}\}$ (which is the K.B.C. for a stationary circle radius **a**) or choose $\mu = 2\pi U a^2$.

Steady flow past a circle (U,a):



$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

$$V_\theta\big|_{r=a} = -2U\sin\theta\begin{cases} = 0 \text{ at } \ \theta = 0, \pi \\ = \mp 2U \text{ at } \ \theta = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$

S_A and $S_B -$ stagnation points.
maximum tangential velocity

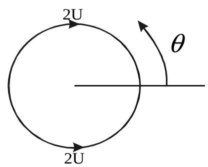
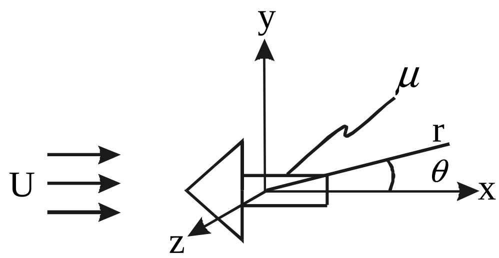


Illustration of the points where the flow reaches maximum speed around the circle.

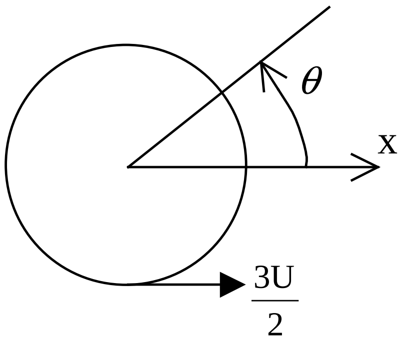
3D:



$$\phi = Ux + \frac{\mu}{4\pi}\frac{\cos\theta}{r^2}, \quad x = r\cos\theta$$
$$V_r = \frac{\partial\phi}{\partial r} = \cos\theta\left(U - \frac{\mu}{2\pi r^3}\right)$$
$$\underbrace{V_r = 0 \text{ on } \; r = a}_{\text{K.B.C. on stationary sphere of radius } a} \rightarrow a = \sqrt[3]{\frac{\mu}{2\pi U}} \text{ or } \; \mu = 2\pi U a^3$$

Steady flow past a sphere (U, a):

$$\phi = \mathrm{U}\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$
$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right)$$
$$V_\theta\,|_{r=a} = -\frac{3U}{2}\sin\theta \begin{cases} = 0 \text{ at } \; \theta = 0, \pi \\ = -\frac{3U}{2} \text{ at } \theta = \frac{\pi}{2} \end{cases}$$



8. **2D corner flow**

Potential function: $\phi = r^\alpha \cos \alpha\theta,$ and the Stream function: $\psi = r^\alpha \sin \alpha\theta$

(a) $\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$

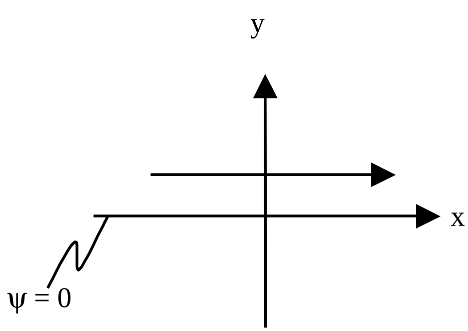
(b)

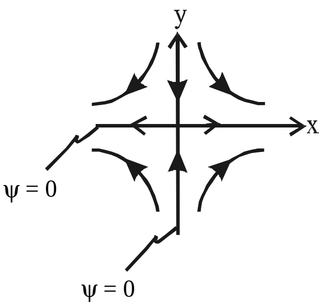
$$u_r = \frac{\partial\phi}{\partial r} = \alpha r^{\alpha-1}\cos\alpha\theta$$
$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\alpha r^{\alpha-1}\sin\alpha\theta$$
$$\therefore u_\theta = 0\,\{\text{ or } \psi = 0\} \text{ on } \alpha\theta = n\pi, n = 0, \pm1, \pm2, \ldots$$

i.e. on $\theta = \theta_0 = 0, \frac{\pi}{\alpha}, \frac{2\pi}{\alpha}, \ldots (\theta_0 \leq 2\pi)$

i. **interior corner flow** – stagnation point origin: $\alpha > 1$.

e.g. $\alpha = 1,\ \theta_0 = 0, \pi, 2\pi, \quad u = 1,\ v = 0$



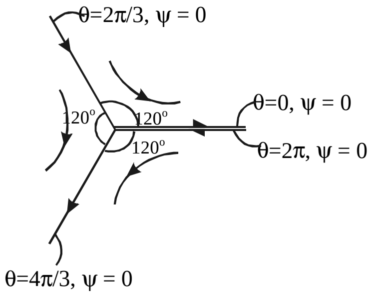


$$\alpha = 2, \quad \theta_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \quad u = 2x, v = -2y$$

(90° corner)

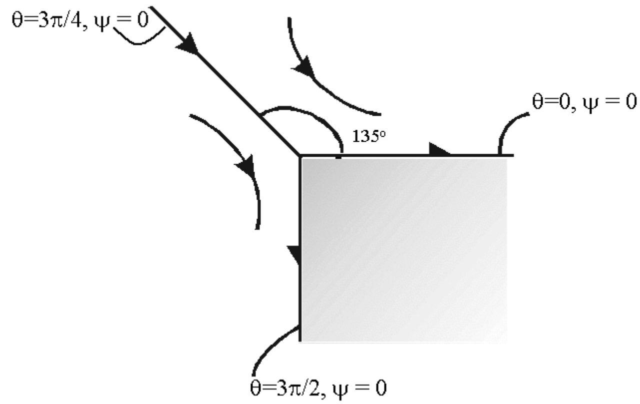
$$\alpha = 3/2, \quad \theta_0 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

(120° corner)



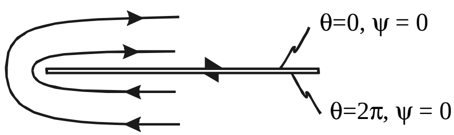
$\alpha = 4/3, \quad \theta_0 = 0, \frac{3\pi}{4}, \frac{3\pi}{2}$ ~~$\frac{9\pi}{4}$~~

(135° corner)

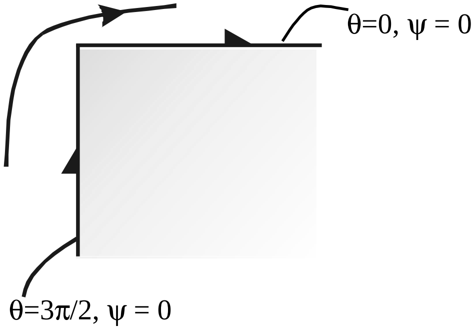


ii. **Exterior corner flow**, $|v| \to \infty$ at origin: $\alpha < 1 (1/2 \le \alpha < 1)$. $\theta_0 = 0, \frac{\pi}{\alpha}$ only. Since we need $\theta_0 \le 2\pi$, we therefore require $\frac{\pi}{\alpha} \le 2\pi$, i.e. $\alpha \ge 1/2$ only.

e.g. $\alpha = 1/2, \theta_0 = 0, 2\pi$ (1/2 infinite plate, flow around a tip)



$\alpha = 2/3, \theta_0 = 0, \frac{3\pi}{2}$ (90^o exterior corner)


θ=0, ψ = 0
θ=3π/2, ψ = 0