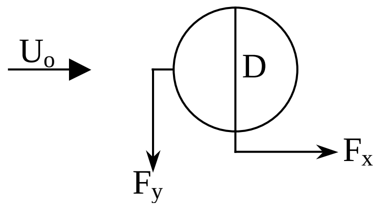
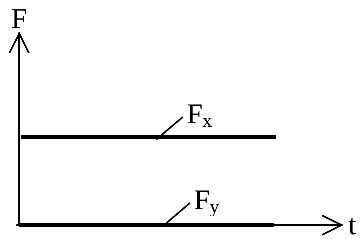
13.021 - Marine Hydrodynamics
Lecture 15

Vortex Shedding and Vortex Induced Vibrations

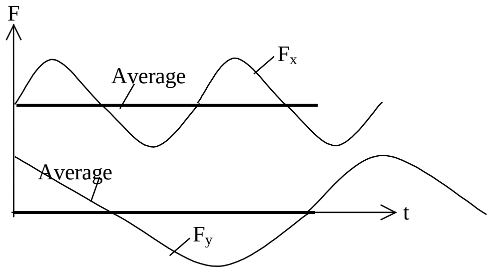
Consider a steady flow U_o on a bluff body with diameter D.

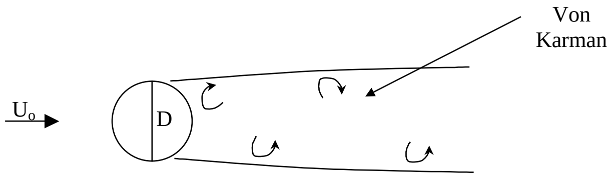


We would **expect** the average forces to be:



The **measured** oscillatory forces are:



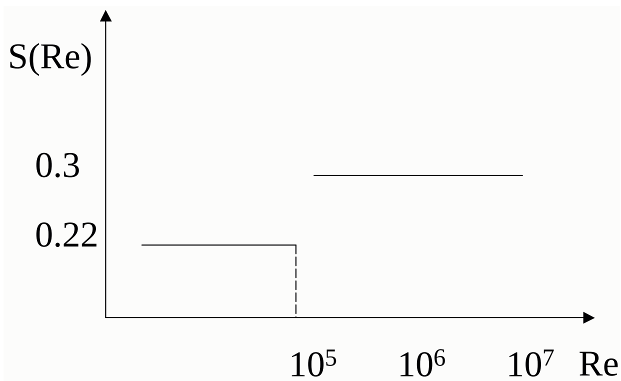


Von Karman Street: Unsteady non-symmetric wake of staggered array of vortices.

Frequency of vortex shedding f $= \omega/2\pi$ is given by a non-dimensional number.

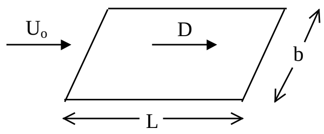
$$\frac{fD}{U_0} = S(Re)$$

where f is the Strouhal frequency, D is the body diameter and S is the Strouhal number. The Drag F_x has frequency $2f$ and non-zero mean value, and the Lift F_y has frequency f, but zero mean value. For laminar flow $S \sim 0.22$ for a cylinder, and for turbulent flow, $S \sim 0.3$ for a cylinder.



C_D and C_L are functions of the correlation length. For "∞" correlation length, $C_L \sim O(1)$ for a fixed cylinder, comparable to C_D. For a moving cylinder, if the Strouhal frequency f is close to one of the cylinder natural frequencies, lock-in occurs. Therefore, if one natural frequency is close to the Strouhal Frequency f_S, we have large amplitude of motion $\Rightarrow$**Vortex induced vibration** (VIV).

4.2 – Drag on a very streamlined body: Flat Plate

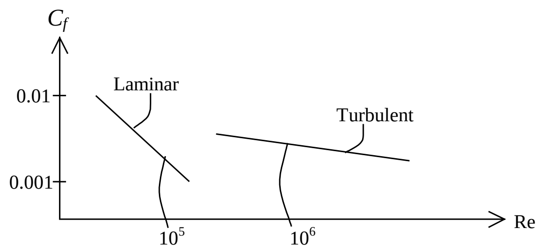


$$\frac{D}{\frac{1}{2}\rho U^2 \underbrace{(Lb)}_{S \text{ (wetted area)}}} = C_f\left(Re, L/b\right)$$

one side of plate

Unlike a bluff body, C_f is a strong function of Re since D is proportional to ν. $\left(\tau = \nu \frac{\partial u}{\partial y}\right)$

Flat Plate Drag



- Re depends on plate smoothness, ambient turbulence, ...

- In general, C_f's are much smaller than C_D's (a factor of $10:100$). Therefore, designing streamlined bodies allows minimal separation and form drag (at the expense of friction drag).
- In general, for streamlined bodies

$$C_{force} \text{ is a combination of } C_D\left(Re\right) \text{ and } C_f\left(Re\right)$$

where C_D is a function of the regime and C_f is a function of Re_L continuously.

Governing equations:

- Navier-Stokes':

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} + \frac{1}{\rho}\vec{f}$$

- Conservation of mass:

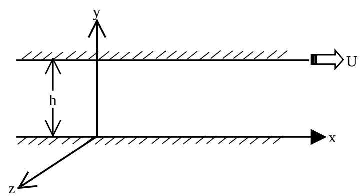
$$\nabla \cdot \vec{v} = 0$$

- Boundary conditions on solid boundaries "no-slip":

$$\vec{v} = \vec{U}$$

Equations very difficult to solve, analytic solution only for a few very special cases (usually when $\left(\vec{v} \cdot \nabla\right)\vec{v} = 0\ldots$)

4.3 Steady Laminar Flow Between 2 Infinite Parallel Walls - Plane Couette Flow



Assume steady flow ($\frac{\partial}{\partial t} = 0$). For the horizontal dimensions $(x, z) >> h$, we assume flow independent of x and z, i.e., $\frac{\partial \vec{v}}{\partial x}, \frac{\partial \vec{v}}{\partial z} = 0$, so $\vec{v} = \vec{v}(y)$.

- Kinematic boundary conditions (k.b.c.):

$$
\begin{aligned}
\vec{v} =& (0,0,0) \text{ on } \; y = 0 \\
\vec{v} =& (U,0,0) \text{ on } \; y = h
\end{aligned}
$$

- Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial v}{\partial y} = 0 \rightarrow v = v(x, z),$$

  but $v = 0$ on $y = 0, h$ (k.b.c.), therefore $v = 0$.

- Navier-Stokes equation for steady flow ($\frac{\partial}{\partial t} = 0$), no $\vec{f}$ and $\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial z} = v = 0$:

$$
\begin{aligned}
u :& \nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial x} \\
v :& \frac{\partial p}{\partial y} = 0 \rightarrow p = p\left(x, z\right) \\
w :& \nu \frac{\partial^2 w}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial z}
\end{aligned}
$$

  We <u>assume</u> that $p = p(x)$, i.e. $\frac{\partial p}{\partial z} \equiv 0$, then $\nu \frac{\partial^2 w}{\partial y^2} = 0 \rightarrow w = a + by$. But k.b.c.: $w = 0$ on $y = 0, h$. Therefore, $w \equiv 0$.

  Finally: $v = w = 0, u = u(y), p = p(x)$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} \quad \text{where } \mu = \rho \nu$$

- Solution:

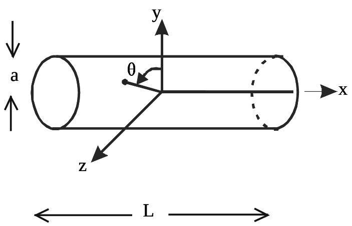
$$u = \tfrac{1}{2} y^2 \tfrac{1}{\mu} \tfrac{\partial p}{\partial x} + C_1 + C_2 y$$

k.b.c.: $C_1 = 0$ and $C_2 = \left(U - \frac{1}{2}h^2\frac{1}{\mu}\frac{dp}{dx}\right)$ since $u(0) = 0$ and $u(h) = U$. Finally,

$$u = \frac{1}{2\mu}(y-h)\,y\frac{dp}{dx} + \frac{Uy}{h} \text{ (plane) } \textbf{Couette flow}$$

4.4 - Steady Laminar Flow in a pipe - Poiseuille Flow.

Assume steady, and for $L >> a$, $\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial \theta} \equiv 0 \rightarrow \vec{v} = \vec{v}(r), r^2 = y^2 + z^2$.



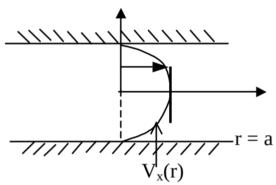
$$\vec{v} = (v_x, v_r, v_\theta)$$

Can show:

$$v_r = v_\theta = 0,\ v_x = v_x(r),\ p = p(x)$$

$$\frac{1}{\rho}\frac{dp}{dx} = \nu \underbrace{\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right)\right)}_{\substack{r \text{ component of } \nabla^2 \\ \text{in cylindrical coordinates}}}$$

K.b.c.: $v_x(a) = 0$ (no slip) and $\frac{dv_x}{dr}(0) = 0$ (symmetry).



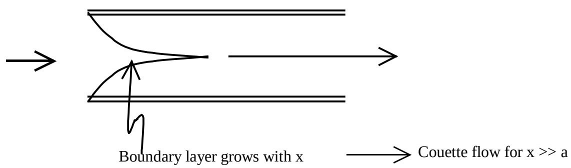
Solution:

$$v_x(r) = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)\left(a^2 - r^2\right) \text{ Pouseuille flow}$$

4.5 Unsteady Flow (boundary layer growth) over an infinite flat plate

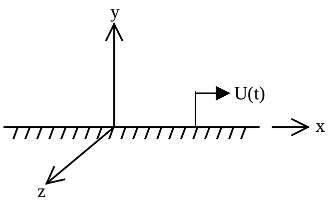
For steady $\begin{pmatrix} Couette \\ Poiseuille \end{pmatrix}$ flow, vorticity, viscosity effects diffuse to all $\begin{pmatrix} h \\ a \end{pmatrix}$

1. limit x



2. limit t (§4.5)

Consider the simplest example of an infinite plate in unsteady motion:



Assuming $\nabla p = 0$, we have $\nabla \frac{\partial \vec{v}}{\partial x}, \frac{\partial \vec{v}}{\partial z} = 0$, so $\vec{v} = \vec{v}\,(y,t)$

Can show that $v = w = 0$ and $u = u(y,t)$.

$$\frac{\partial u}{\partial t} + u \underbrace{\frac{\partial u}{\partial x}}_{=0} + \underbrace{v}_{=0} \frac{\partial u}{\partial y} + w \underbrace{\frac{\partial u}{\partial z}}_{=0} = -\frac{1}{\rho} \underbrace{\frac{\partial p}{\partial x}}_{=0} + \nu \left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{=0} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{=0} \right)$$

Finally:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \underbrace{\text{“heat”}}_{\substack{\text{momentum} \\ \text{(velocity)}}} \quad \text{diffusion equation} \tag{1}$$

B.c.: $u(0, t) = U(t), t > 0$; u bounded ($\rightarrow 0$) as $y \rightarrow \infty$ + suitable initial condition.

4.5.1 Sinusoidally Oscillating Plate

$U(t) = U_o \cos \omega t = \text{Real}\left\{U_o e^{i\omega t}\right\}$ $\quad e^{i\alpha} = \cos \alpha + i \sin \alpha$ where α is real. Let $u(y, t) = \text{Real}\left\{f(y) e^{i\omega t}\right\}$ where f(y) is an unknown complex (magnitude & phase) amplitude. Then (1):

$$i\omega f = \nu \frac{d^2 f}{dy^2} \leftarrow 2^{nd} \text{ order ODE for} f(y)$$

General Solution:

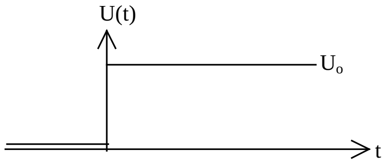
$$f(y) = C_1 e^{(1+i)\left(\sqrt{\omega/2\nu}\right) y} + C_2 e^{-(1+i)\left(\sqrt{\omega/2\nu}\right) y}$$

B.c.: $u \rightarrow$ bounded as $y \rightarrow \infty$, $C_1 = 0$. $u \rightarrow U(t)$ as $y = 0, C_2 = U_o$.

Finally:

$$u(y, t) = U_o e^{-\left(\sqrt{\omega/2\nu}\right) y} \cos\left(-\sqrt{\frac{\omega}{2\nu}} y + \omega t \right) \quad \textbf{Stokes' (Oscillatory) b.l.}$$

4.5.2 Impulsively Started Plate



$$u\left(y,t\right):\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$$

B.c.:

$$\left.\begin{array}{l}u(o,t)=U_o\\u(\infty,t)=0\end{array}\right\}\text{ for } t>0, \text{i.e. } u(y,0)=0$$

Problem has no explicit time scale, can use dimensional analysis to solve in terms of a similarity parameter:

$$\frac{u}{U_o}=f\left(y,t,\nu\right)=\underbrace{f\left(\frac{y}{2\sqrt{\nu t}}\right)}_{\equiv\eta \text{ similarity parameter}};\text{i.e. }\frac{u}{U_o}=\underbrace{f\left(\eta\right)}_{\text{Self similar solution}}$$

Solution:

$$\underbrace{\frac{u}{U_o}}_{\text{Impulsively started flat-plate boundary layer solution}}=\underbrace{erfc\left(\eta\right)}_{\text{Complementary error function}}=1-erf\left(\eta\right)=1-\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\alpha^2}d\alpha$$