13.021 - Marine Hydrodynamics
Lecture 17

4.7 – Turbulent Flow – Reynolds Stress

$$u = \bar{u} + u' \tag{1}$$

By definition:

$$\bar{u'} = \overline{u - \bar{u}} = \bar{u} - \bar{u} = 0, \text{ also } \frac{\partial}{\partial x}\bar{u} = \overline{\frac{\partial u}{\partial x}} \text{ etc.}$$

Substitute Eq. (1) into governing equations and take $\overline{(\quad)}$:

$$\overline{\text{continuity}}: \quad \overline{\frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial \bar{u}_i}{\partial x_i}} + \underbrace{\overline{\frac{\partial u_i'}{\partial x_i}}}_{\to 0} = 0, \qquad \therefore \frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\text{but} \quad \frac{\partial u_i}{\partial x_i} = 0 = \underbrace{\frac{\partial \bar{u}_i}{\partial x_i}}_{\to 0 \text{ just shown}} + \frac{\partial u_i'}{\partial x_i}, \qquad \therefore \frac{\partial u_i'}{\partial x_i} = 0$$

Momentum Equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

We write $u_i = \bar{u}_i + u_i$ and we apply $\overline{(\quad)}$ then:

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial \bar{u}_i}{\partial t} + \underbrace{\overline{\frac{\partial u'_i}{\partial t}}}_{0}; \text{ similarly } \begin{cases} \overline{\nu \nabla^2 u_i} = \nu \nabla^2 \overline{u_i} \\ \overline{\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i} \overline{(\bar{p} + p')} = \frac{\partial \bar{p}}{\partial x_i} \text{ etc.} \end{cases}$$

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\left(\bar{u}_j + u'_j\right) \frac{\partial}{\partial x_j} \left(\bar{u}_i + u'_i\right)} = \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \underbrace{\overline{u'_j \frac{\partial \bar{u}_i}{\partial x_j}}}_{0} + \underbrace{\overline{\bar{u}_j \frac{\partial u'_i}{\partial x_j}}}_{0} + \overline{u'_j \frac{\partial}{\partial x_j} u'_i}$$

From continuity:

$$\overline{u'_j \frac{\partial}{\partial x_j} u'_i} = \frac{\partial}{\partial x_j} \overline{u'_j u'_i} - \overline{u'_i \underbrace{\frac{\partial u'_j}{\partial x_j}}_{0 \rightarrow \text{by continuity}}}$$

Finally:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \underbrace{-\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 \bar{u}_i}_{\frac{1}{\rho} \frac{\partial}{\partial x_j} \overline{\tau_{ij}}} - \frac{\partial}{\partial x_j} \overline{u'_i u'_j}$$

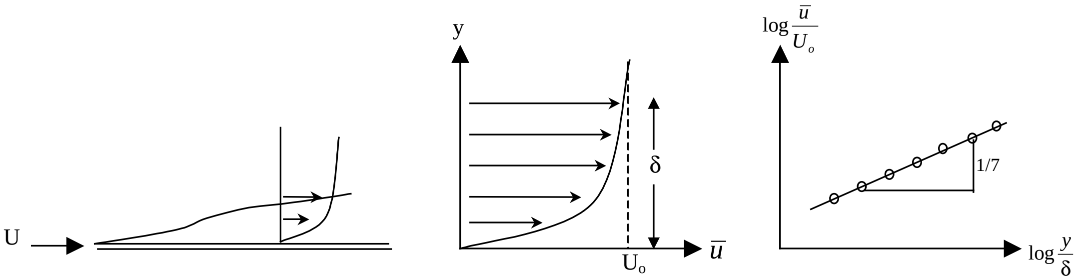
Reynolds averaged N-S equation:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\overline{\tau_{ij}} - \rho \overline{u'_i u'_j}\right]$$

Reynolds stress:

$$\tau_{R_{ij}} \equiv -\rho \overline{u'_i u'_j}$$

4.8 – Turbulent Boundary Layer over a Smooth Flat Plate.



(1/7)th power velocity profile law:

$$\frac{\bar{u}}{U_o} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}, \tag{2}$$

where $\delta = \delta(x)$ to be determined. From equation (2):

$$\delta^* = \frac{\delta}{8}$$

$$\theta = \frac{7}{72}\delta \cong 0.0972\ \delta$$

$$\frac{\tau_o}{\rho U_o^2} = 0.0227\left(\frac{U_o\delta}{\nu}\right)^{-\frac{1}{4}} \leftarrow$$ using another empirical formula for friction (Blasius' law of friction) for pipes

Von Karman's moment equation:

$$\frac{\tau_o}{\rho U_o^2} = \frac{d}{dx}(\theta) \rightarrow 0.0227\left(\frac{U_o\delta}{\nu}\right)^{-\frac{1}{4}} = \frac{7}{72}\frac{d\delta}{dx} \leftarrow \text{ ODE for } \delta$$

$$\frac{\delta}{x} \cong 0.373 R_x^{-\frac{1}{5}}$$

$$\delta(x) \cong 0.373x\left(\frac{U_o x}{\nu}\right)^{-\frac{1}{5}}$$

For $\delta(0) = 0$ and assuming turbulent boundary layer at $x = 0$, i.e., tripped at $x = 0$ or $R_x >> 1$,

$$\frac{\delta}{x} \cong 0.373 R_x^{-\frac{1}{5}}$$

Then,

$$\delta(x) \sim \sqrt{x} \text{ laminar}$$
$$\delta(x) \sim x^{4/5} \text{ turbulent (grows much faster)}$$

	Laminar	Turbulent
Blasius	$\delta^* \sim 1.72\sqrt{\frac{\nu}{U_o}}\sqrt{x}$	$\delta^* \sim 0.047\left(\frac{\nu}{U_o}\right)^{\frac{1}{5}} x^{4/5} \ldots \left(\frac{1}{7}\right)$ th power law

$$D = 0.036\left(\rho U_o^2\right) BL\ R_L^{-1/5}$$

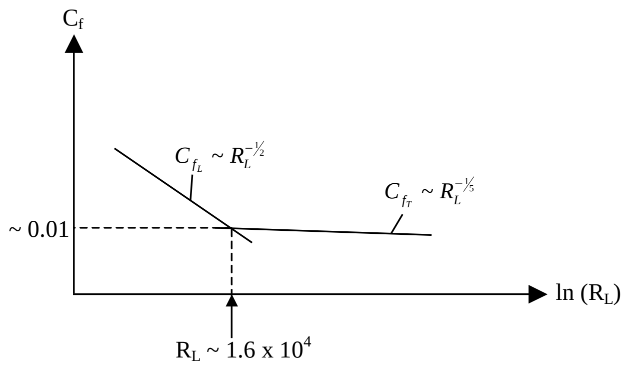
$$C_f = \frac{\mathrm{D}}{\frac{1}{2}\rho U_o^2 BL} = 0.073 R_L^{-1/5} \text{ for } R_L > 5 \times 10^5$$

Logarithmic Velocity Profile Law.

$$\frac{0.242}{\sqrt{C_f}} = \log_{10}\left(R_L C_f\right) \ \leftarrow \ \text{Schoenherr's formula}$$

Summary of Boundary Layer over a Flat Plate.

Laminar (Blasius')	**Turbulent (1/7 power law)**
$\frac{\delta}{x} \sim R_x^{-1/2}$ $\delta^* \cong 1.72 \times R_x^{-1/2} \sim \sqrt{x}$	$\frac{\delta}{x} \sim R_x^{-1/5}$ $\delta^* = \frac{\delta}{8} \cong 0.047 x R_x^{-1/5} \sim x^{4/5}$
$\tau_o \cong 0.332 \rho U_o^2 R_x^{-1/2}$	$\tau_o \cong 0.0227 \rho U_o^2 R_\delta^{-1/4}$ $\cong 0.029 \rho U_o^2 R_x^{-1/5}$
$D \cong 0.664 \rho U_o^2 (BL) R_L^{-1/2}$	$D \cong 0.03625 \rho U_o^2 (BL) R_L^{-1/5}$
$C_f = \frac{D}{\rho U_o^2 (BL)}$	
$C_f = 1.328 R_L^{-1/2}$	$C_f = 0.0725 R_L^{-1/5}$



For τ_o, the cross-over is at $R_x \sim 3.4 \times 10^3$, i.e.,

$$(\tau_o)_{\text{laminar}} > (\tau_o)_{\text{turbulent}} \text{ for } R_x < 3.4 \times 10^3$$
$$(\tau_o)_{\text{laminar}} \sim (\tau_o)_{\text{turbulent}} \text{ for } R_x \sim 3.4 \times 10^3$$
$$(\tau_o)_{\text{laminar}} < (\tau_o)_{\text{turbulent}} \text{ for } R_x > 3.4 \times 10^3$$

Therefore, for most prototype scales:

$$(C_f)_{\text{turbulent}} > (C_f)_{\text{laminar}}$$
$$(\tau_o)_{\text{turbulent}} > (\tau_o)_{\text{laminar}}$$