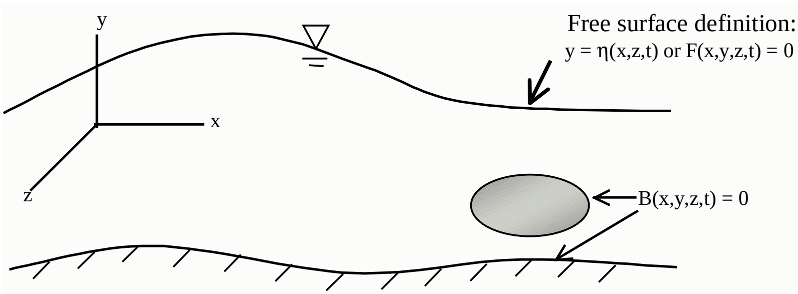
13.021 - Marine Hydrodynamics
Lecture 19

Water Waves

Exact (nonlinear) governing equations for surface gravity waves assuming potential theory



Unknown variables:

Velocity field: $\vec{v}(x,y,z,t) = \nabla\phi(x,y,z,t)$

Position of free surface: $\eta(x,z,t)$ or $F(x,y,z,t)$

Pressure field: $p(x,y,z,t)$

Field equation:

Continuity: $\nabla^2\phi = 0 \quad y < \eta$ or $F < 0$

Given ϕ can calculate p: $\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p-p_a}{\rho} + gy = 0; \quad y < \eta$ or $F < 0$

Far way, no disturbance: $\partial\phi/\partial t,\ \nabla\phi \to 0$ and $p = \underbrace{p_a}_{\text{atmospheric}} - \underbrace{\rho g y}_{\text{hydrostatic}}$

Boundary Conditions:

1. On an impervious boundary $B(x, y, z, t) = 0$, we have KBC:

$$\vec{v} \cdot \hat{n} = \nabla\phi \cdot \hat{n} = \frac{\partial\phi}{\partial n} = \vec{U}\left(\vec{x}, t\right) \cdot \hat{n}\left(\vec{x}, t\right) = U_n \text{ on } B = 0$$

Alternatively: a particle P on B remains on B, i.e. B is a material surface; e.g. if P is on B at $t = t_0$, i.e.

$$B(\vec{x}_P, t_0) = 0, \text{ then } B(\vec{x}_P(t), t_0) = 0 \text{ for all } \; t,$$

so that, following P, $B = 0$.

$$\therefore \frac{DB}{Dt} = \frac{\partial B}{\partial t} + (\nabla\phi \cdot \nabla) B = 0 \text{ on } B = 0$$

For example, flat bottom at $y = -h$:

$$\partial\phi/\partial y = 0 \text{ on } y = -h \text{ or } B : y + h = 0$$

2. On the free surface, $y = \eta$ or $F = y - \eta(x, z, t) = 0$ we have KBC and DBC.

KBC: free surface is a material surface, no perpendicular relative velocity to free surface, particle on free surface remains on free surface:

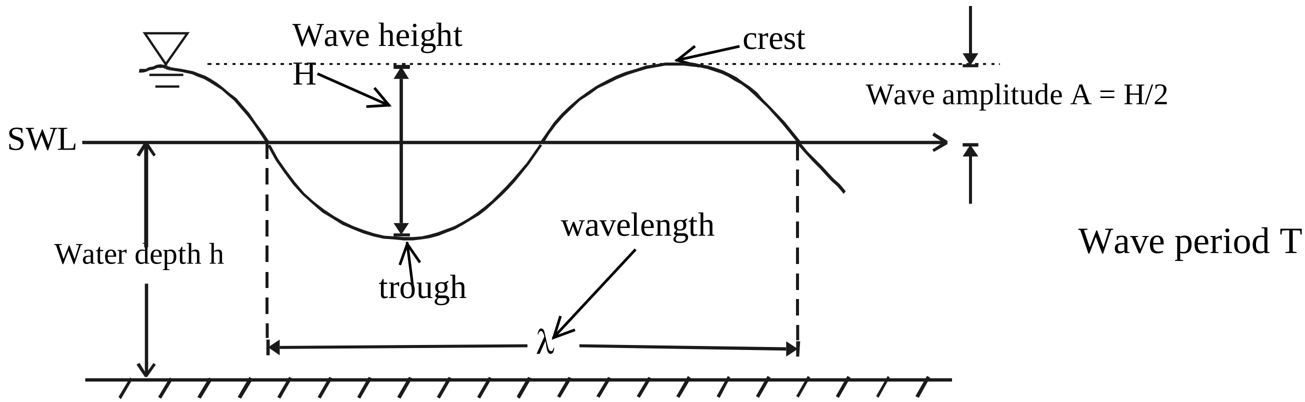
$$\frac{DF}{Dt} = 0 = \frac{D}{Dt}(y - \eta) = \underbrace{\frac{\partial\phi}{\partial y}}_{\substack{\text{vertical} \\ \text{velocity}}} - \frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial x} \underbrace{\frac{\partial\eta}{\partial x}}_{\substack{\text{slope} \\ \text{of f.s.}}} - \frac{\partial\phi}{\partial z} \underbrace{\frac{\partial\eta}{\partial z}}_{\substack{\text{slope} \\ \text{of f.s.}}} \text{ on } y = \underbrace{\eta}_{\substack{\text{still} \\ \text{unknown}}}$$

DBC: $p = p_a$ on $y = \eta$ or $F = 0$. Apply Bernoulli equation at $y = \eta$:

$$\frac{\partial\phi}{\partial t} + \underbrace{\frac{1}{2}|\nabla\phi|^2}_{\text{non-linear term}} + g \underbrace{\eta}_{\text{still unknown}} = 0 \text{ on } y = \eta \text{ (assuming } \; p_a = 0)$$

Linearized (Airy) Wave Theory

Consider small amplitude waves: (small free surface slope)

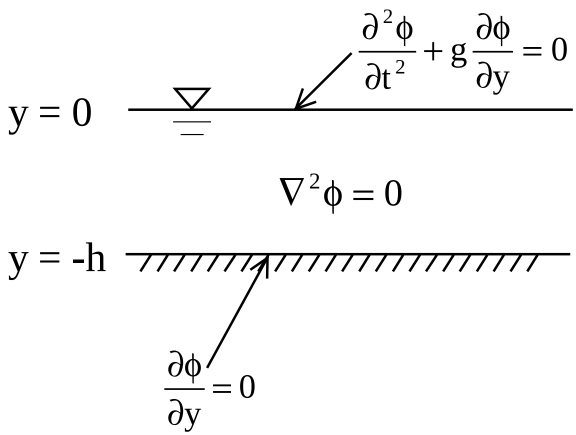


Assume amplitude small compared to wavelength, i.e., $\dfrac{A}{\lambda} << 1$.

Consequently: $\dfrac{\phi}{\lambda^2/T}, \dfrac{\eta}{\lambda} << 1$, and we keep only linear terms in ϕ, η.

$$\text{For example: } ()|_{y=\eta} = \underbrace{()_{y=0}}_{\text{keep}} + \underbrace{\eta \frac{\partial}{\partial y} ()|_{y=0}}_{\text{discard}} + \ldots \text{ Taylor series}$$

Finally the boundary value problem is described by:



Linear (Airy) Waves	
Constant finite depth h	**Infinite depth**
(1) GE: $\nabla^2\phi = 0, \quad -h < y < 0$	$\nabla^2\phi = 0, \quad y < 0$
(2) BKBC: $\frac{\partial \phi}{\partial y} = 0, \quad y = -h$	$\nabla\phi \to 0, \quad y \to -\infty$
(3a) FSKBC: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad y = 0$ **(3b) FSDBC:** $\frac{\partial \phi}{\partial t} + g\eta = 0, \quad y = 0$	**(3)** $\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0, \quad y = 0$

$$\begin{aligned}
&\textbf{Constant finite depth } h && \{\textbf{Infinite depth}\} & \\
&\nabla^2\phi = 0, \quad -h < y < 0 && \left\{\nabla^2\phi = 0, \quad y < 0\right\} & (1)\\
&\frac{\partial\phi}{\partial y} = 0, \quad y = -h && \{\nabla\phi \to 0, \quad y \to -\infty\} & (2)\\
&\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, \quad y = 0 && \left\{\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, \quad y = 0\right\} & (3)
\end{aligned}$$

Given ϕ :

$$\eta(x,t) = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=0} \qquad \left\{\eta(x,t) = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=0}\right\} \tag{4}$$

$$p - p_a = \underbrace{-\rho\frac{\partial\phi}{\partial t}}_{\text{dynamic}} - \underbrace{\rho g y}_{\text{hydrostatic}} \qquad \left\{p - p_a = \underbrace{-\rho\frac{\partial\phi}{\partial t}}_{\text{dynamic}} - \underbrace{\rho g y}_{\text{hydrostatic}}\right\} \tag{5}$$

<u>Solution of 2D Periodic Plane Progressive Waves</u>(using separation of variables)

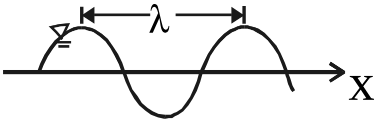
Math . . . (solve (1), (2), (3)). Answer:

$$\phi = \frac{gA}{\omega}\sin(kx - \omega t)\frac{\cosh k(y+h)}{\cosh kh} \qquad \left\{\phi = \frac{gA}{\omega}\sin(kx - \omega t)\, e^{ky}\right\}$$

$$\eta \underbrace{=}_{\text{using (4)}} A\cos(kx - \omega t) \qquad \left\{\eta \underbrace{=}_{\text{using(4)}} A\cos(kx - \omega t)\right\}$$

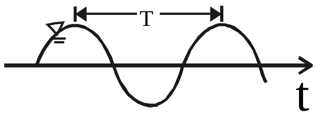
where A is the wave amplitude $= H/2$.

1. At $t = 0$ (say), $\eta = A\cos kx \rightarrow$ periodic in x with wavelength: $\lambda = 2\pi/k$ units of $\lambda :$ $[L]$



k = wavenumber = 2π/λ [L^{-1}]

2. At $x = 0$ (say), $\eta = A\cos \omega t \rightarrow$ periodic in t with period: $T = 2\pi/\omega$ units of $T :$ $[T]$



ω = frequency = 2π/T [T^{-1}], e.g. rad/sec

3. $\eta = A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]$ units of $\frac{\omega}{k}$: $\left[\frac{L}{T}\right]$

Following a point with velocity $\dfrac{\omega}{k}$, i.e. $x_p = \left(\dfrac{\omega}{k}\right)t + const.$ the phase of η does not change:

$\dfrac{\omega}{k} = \dfrac{\lambda}{T} \equiv V_p \equiv$ phase velocity.

Dispersion Relationship

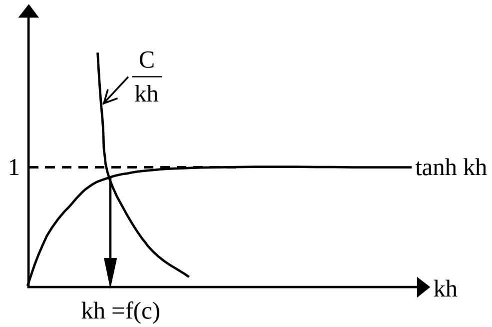
So far, any ω, k combination is allowed. Must satisfy FSBC, substitute ϕ into equation (3):

$$-\omega^2 \cosh kh + gk \sinh kh = 0,$$

which gives:

$$\underbrace{\omega^2 = gk \tanh kh \qquad\qquad\qquad \{\omega^2 = gk\}}_{\text{Dispersion Relationship}} \tag{6}$$

Given k (and h) $\rightarrow \omega$.
Given ω (and h), ...
Dispersion relationship (6) uniquely relates ω and k, given h: $\omega = \omega(k; h)$ or $k = k(\omega; h)$



$$C \equiv \frac{\omega^2 h}{g} \underbrace{=}_{\text{from (6)}} (kh) \tanh(kh)$$

$$\frac{C}{kh} = \tanh kh$$

$$\rightarrow \text{ obtain unique solution for } k$$

In general: ($k \uparrow$ as $\omega \uparrow$) or ($\lambda \uparrow$ as $T \uparrow$).

$$\frac{\lambda}{T} = V_p = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kh} \qquad\qquad \left\{ V_p = \sqrt{\frac{g}{k}} \right\}$$

Also: $V_p \uparrow$ as $T \uparrow$ or $\lambda \uparrow$.

For fixed k (or λ), $V_p \uparrow$ as $h \uparrow$ (from shallow to deeper water)

$V_p = V_p(k)$ or $V_p(\omega) \rightarrow$ frequency dispersion

Solution of the Dispersion Relationship : $\omega^2 = gk \tanh kh$

Property of $\tanh kh$:

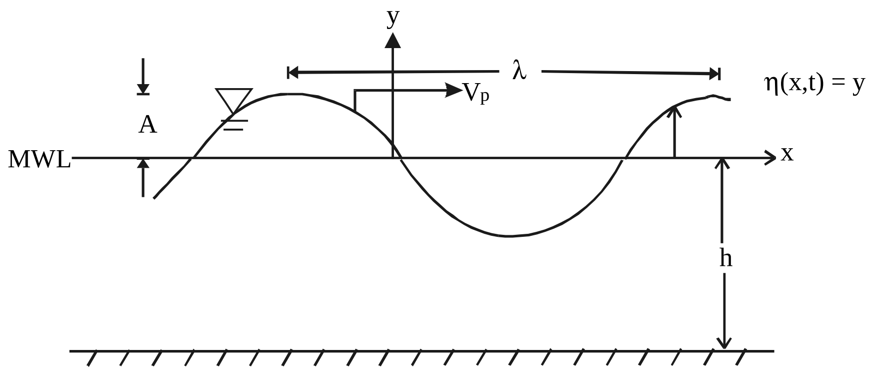
$$\tanh kh = \frac{\sinh kh}{\cosh kh}$$
$$= \frac{1-e^{-2kh}}{1+e^{-2kh}} \cong \begin{cases} kh \text{ for } kh << 1, \text{ i.e. } h << \lambda & \text{(long waves or shallow water)} \\ 1 \quad \text{for } kh >\sim 3, \text{ i.e. } kh > \pi \rightarrow h > \frac{\lambda}{2} & \text{(short waves or deep water)} \text{(e.g. } \tanh 3 = 0.995\text{)} \end{cases}$$

Deep water waves or short waves	**Intermediate depth or wavelength**	**Shallow water waves or long waves**
$kh >> 1$ $(h >\sim \lambda/2)$	Need to solve $\omega^2 = gk \tanh kh$ given ω, h for k (given k, h for ω - easy!)	$kh << 1$ $(h/\lambda <\sim 1/20$ in practice)
$\omega^2 = gk$ $\lambda = \frac{g}{2\pi}T^2$ $\left(\lambda(\text{in ft.}) \approx 5.12T^2 \text{ (in sec.)}\right)$ $V_p = \frac{\omega}{k} = \frac{g}{\omega} = \sqrt{\frac{g}{2\pi}\lambda}$ **Frequency dispersion** $\lambda = \frac{2\pi}{g}V_p^2$	(a) Use tables or graphs (e.g.JNN fig.6.3) $\omega^2 \quad = gk \tanh kh = gk_\infty$ $\Rightarrow \frac{k_\infty}{k} = \frac{\lambda}{\lambda_\infty} = \frac{V_p}{V_{p\infty}} = \tanh kh$ (b) Use numerical approximation (hand calculator, about 4 decimals) i. Calculate $C = \omega^2 h/g$ ii. If $C > 2$: "deeper" $\Rightarrow$ $kh \approx C(1 + 2e^{-2C} - 12e^{-4C} + \ldots)$ If $C < 2$: "shallower" $\Rightarrow$ $kh \approx \sqrt{C}(1 + 0.169C + 0.031C^2 + \ldots)$	$\omega^2 \cong gk \cdot kh \rightarrow \omega = \sqrt{gh}\ k$ $\lambda = \sqrt{gh}\ T$ $V_p = \frac{\omega}{k} = \frac{\lambda}{T} = \sqrt{gh}$ **No frequency dispersion** $\frac{\omega}{k} = \sqrt{gh}$

Characteristics of a Linear Plane Progressive Wave

$$k = \frac{2\pi}{\lambda}$$
$$\omega = \frac{2\pi}{T}$$
$$H = 2A$$

Define $U \equiv \omega A$



Linear Solution:

$$\phi = \frac{Ag}{\omega}\frac{\cosh k\left(y+h\right)}{\cosh kh}\sin\left(kx-\omega t\right); \qquad \eta = A\cos\left(kx-\omega t\right)$$

with:

$$\omega^2 = gk\tanh kh$$

Velocity field:

$u = \frac{\partial \phi}{\partial x} = \frac{Agk}{\omega}\frac{\cosh k\left(y+h\right)}{\cosh kh}\cos\left(kx-\omega t\right) = \underbrace{A\omega}_{U}\frac{\cosh k\left(y+h\right)}{\sinh kh}\cos\left(kx-\omega t\right)$	$v = \frac{\partial \phi}{\partial y} = \frac{Agk}{\omega}\frac{\sinh k\left(y+h\right)}{\cosh kh}\sin\left(kx-\omega t\right) = \underbrace{A\omega}_{U}\frac{\sinh k\left(y+h\right)}{\sinh kh}\sin\left(kx-\omega t\right)$

On $y=0$:

$u = U_o = A\omega\frac{1}{\tanh kh}\cos\left(kx-\omega t\right)$	$v = V_o = A\omega\sin\left(kx-\omega t\right) = \frac{\partial \eta}{\partial t}$
$\frac{u}{U_o} = \frac{\cosh k\left(y+h\right)}{\cosh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1 & \text{shallow water}\end{cases}$	$\frac{v}{V_o} = \frac{\sinh k\left(y+h\right)}{\sinh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1+\frac{y}{h} & \text{shallow water}\end{cases}$
• u is in phase with η	• v is out of phase with η

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<tr><th>Shallow water</th><th>Intermediate water</th><th>Deep water</th></tr>
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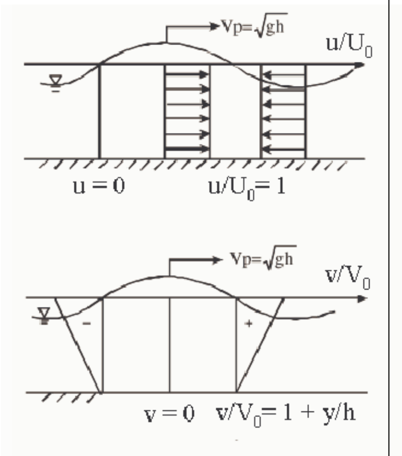
<td>
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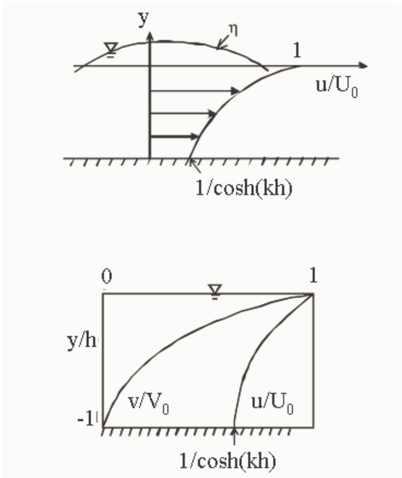
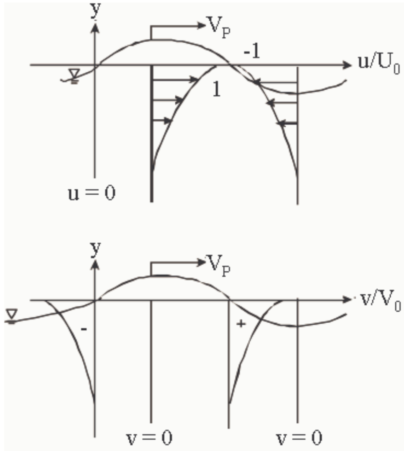
<td>

Shallow water / Long waves: kh << 1
$u = \frac{A\omega}{kh}\cos(kx - \omega t) = \eta\sqrt{\frac{g}{h}}$
$v = A\omega\left(1 + \frac{y}{h}\right)\sin(kx - \omega t)$</td>
<td>

Rule of thumb $\frac{u}{u_o} = \frac{v}{v_o} \approx 4\%$ at $y = -\frac{\lambda}{2}$
$(\cosh\ kh \sim 1, \sinh\ kh \sim kh)$</td>
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<tr><td colspan="3"><u>Pressure Field:</u> dynamic pressure $p_d = -\rho\frac{\partial\phi}{\partial t}$; total pressure $p = p_d - \rho g y$</td></tr>
<tr>
<td>$p_d = \rho g \eta \quad$ (no decay)

$p = \underbrace{\rho g(\eta - y)}_{\text{“hydrostatic” approximation}}$</td>
<td>$$p_d = \rho g A \frac{\cosh k(y+h)}{\cosh kh}\cos(kx - \omega t)$$
$$= \rho g \frac{\cosh k(y+h)}{\cosh kh}\eta$$
$\frac{p_d}{p_{d_o}}$ same picture as $\frac{u}{U_o}$
$$\frac{p_d(-h)}{p_{d_o}} = \frac{1}{\cosh kh}$$</td>
<td>$p_d = \rho g e^{ky}\eta$

$$\frac{p_d(-h)}{p_{d_o}} = e^{-ky}$$
$$p = \rho g\left[\eta e^{ky} - y\right]$$</td>
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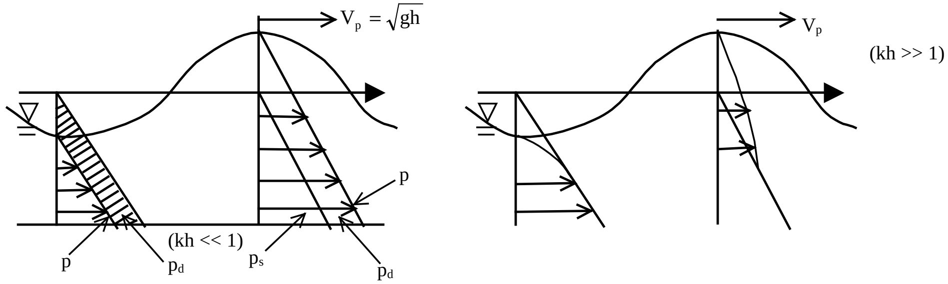
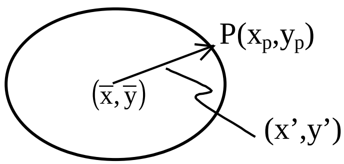


Particle Orbit/ Velocity (Lagrangian).

Let $x_p(t), y_p(t)$ be the position of the particle P, then $x_p(t) = \bar{x} + x'(t); y_p(t) = \bar{y} + y'(t)$ where $(\bar{x}; \bar{y})$ is the mean position of P.



$$v_p \approx v\left(\bar{x}, \bar{y}, t\right)\left(\bar{x}, \bar{y}\right)$$

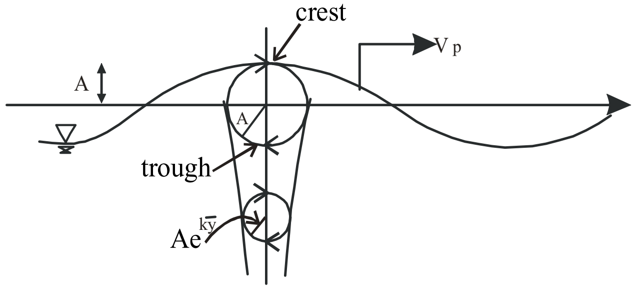
$$u_p = \frac{dx_p}{dt} = u\left(\bar{x}, \bar{y}, t\right) + \underbrace{\frac{\partial u}{\partial x}\left(\bar{x}, \bar{y}, t\right) x' + \frac{\partial u}{\partial y}\left(\bar{x}, \bar{y}, t\right) y' + \ldots}_{\text{ignore - linear theory}}$$

$$\begin{aligned}
x_p &= \bar{x} + \int dt\; u\left(\bar{x}, \bar{y}, t\right) = \bar{x} + \int dt\; \omega A \frac{\cosh k\left(\bar{y}+h\right)}{\sinh kh} \cos\left(k\bar{x} - \omega t\right) \\
x' &= -A \frac{\cosh k\left(\bar{y}+h\right)}{\sinh kh} \sin\left(k\bar{x} - \omega t\right) \\
y' &= \int dt\; v\left(\bar{x}, \bar{y}, t\right) = \int dt\; \omega A \frac{\sinh k\left(\bar{y}+h\right)}{\sinh kh} \sin\left(k\bar{x} - \omega t\right) \\
&= A \frac{\sinh k\left(\bar{y}+h\right)}{\sinh kh} \cos\left(k\bar{x} - \omega t\right) \;\text{ on } \bar{y} = 0, y' = A\cos\left(k\bar{x} - \omega t\right) = \eta
\end{aligned}$$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \text{ or } \frac{\left(x_p - \bar{x}\right)^2}{a^2} + \frac{\left(y_p - \bar{y}\right)^2}{b^2} = 1$$

where $a = A\frac{\cosh k(\bar{y}+h)}{\sinh kh}; b = A\frac{\sinh k(\bar{y}+h)}{\sinh kh}$, i.e. the particle orbits form closed ellipses with horizontal and vertical axes a and b.

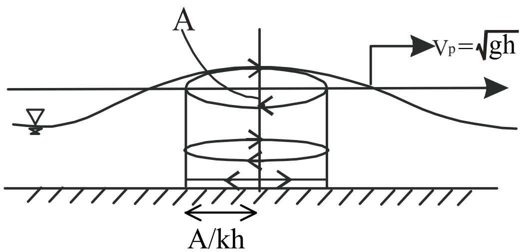
(a) deep water $kh >> 1$: $a = b = Ae^{k\bar{y}}$
circular orbits with radii $Ae^{k\bar{y}}$ decreasing
exponentially with depth



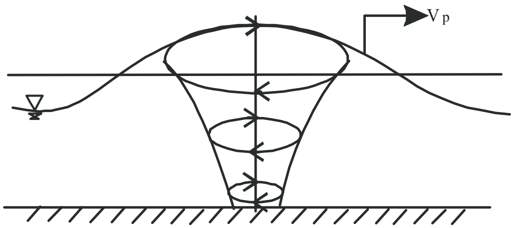
(b) shallow water $kh << 1$:

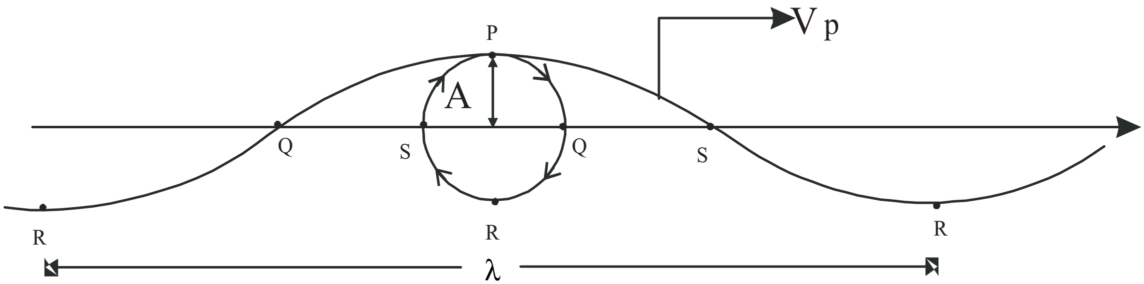
$a = \frac{A}{kh} =$ const. ; $b = A(1+\frac{y}{h})$

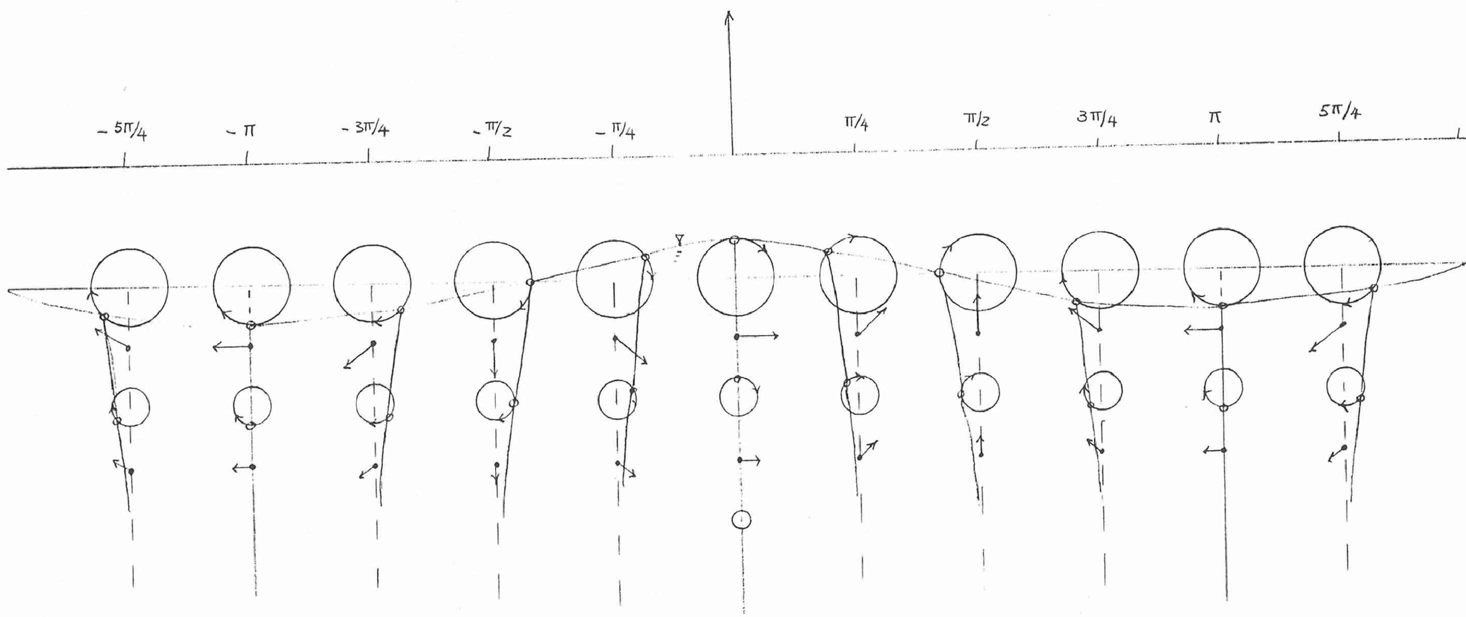
decreases linearly
with depth



(c) Intermediate depth







- 5π/4
- π
- 3π/4
- π/2
- π/4
π/4
π/2
3π/4
π
5π/4

Summary of Plane Progressive Wave Characteristics

$f(y)$	Deep water/ short waves $kh > \pi$ (say)	Shallow water/ long waves $kh << 1$
$\frac{\cosh k(y+h)}{\cosh kh} = f_1(y) \sim$ e.g.p_d	e^{ky}	1
$\frac{\cosh k(y+h)}{\sinh kh} = f_2(y) \sim$ e.g.u, a	e^{ky}	$\frac{1}{kh}$
$\frac{\sinh k(y+h)}{\sinh kh} = f_3(y) \sim$ e.g. v, b	e^{ky}	$1 + \frac{y}{h}$

$C\left(x\right)=\cos\left(kx-\omega t\right)$ (in phase with η)	$S\left(x\right)=\sin\left(kx-\omega t\right)$ (out of phase with η)
$\frac{\eta}{A}=C\left(x\right)$	
$\frac{u}{A\omega}=C\left(x\right)f_2\left(y\right)$	$\frac{v}{A\omega}=S\left(x\right)f_3\left(y\right)$
$\frac{p_d}{\rho gA}=C\left(x\right)f_1\left(y\right)$	
$\frac{y'}{A}=C\left(x\right)f_3\left(y\right)$	$\frac{x'}{A}=-S\left(x\right)f_2\left(y\right)$
$\frac{a}{A}=f_2\left(y\right)$	$\frac{b}{A}=f_3\left(y\right)$