13.021 – Marine Hydrodynamics, Fall 2004
Lecture 20

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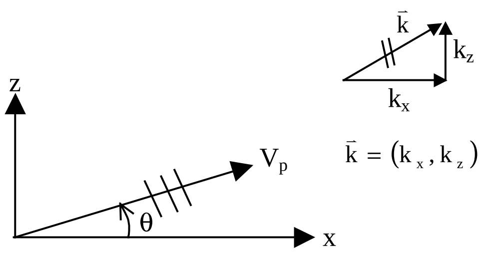
Free-surface waves: linear superposition, group velocity and wave energy

Superposition of linear plane progressive waves

1. Oblique Plane Waves:



(Looking up the y-axis from below the surface)

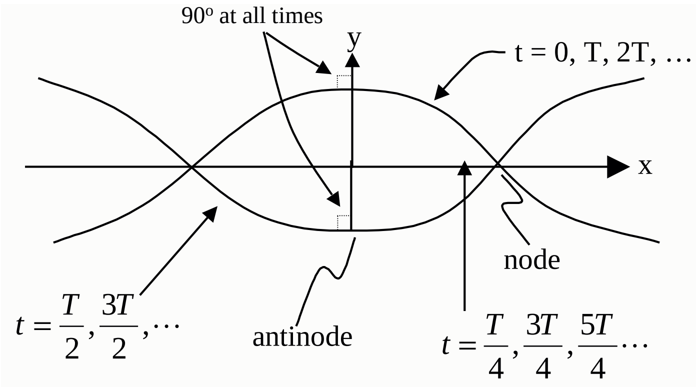
Consider wave propagation at an angle θ to the x-axis

$$\begin{aligned}
\eta =& A\cos(\overbrace{kx\cos\theta + kz\sin\theta}^{\vec{k}\cdot\vec{x}} - \omega t) = A\cos\left(k_x x + k_z z - \omega t\right)\\
\phi =& \frac{gA}{\omega}\frac{\cosh k\left(y+h\right)}{\cosh kh}\sin\left(kx\cos\theta + kz\sin\theta - \omega t\right)\\
\omega^2 =& gk\tanh kh; k_x = k\cos\theta, k_z = k\sin\theta, k = \sqrt{k_x^2 + k_z^2}
\end{aligned}$$

2. <u>Standing Waves:</u>

Same A, k, ω, no phase shift

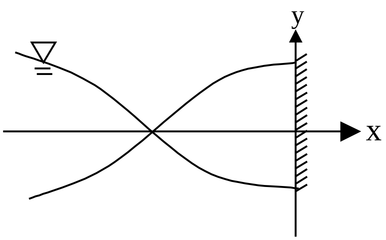
$$\eta = A\cos(kx-\omega t) + A\cos(-kx-\omega t) = 2A\cos kx\cos\omega t$$
$$\phi = -\frac{2gA}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\cos kx\sin\omega t$$



$$\frac{\partial\eta}{\partial x} \sim \frac{\partial\phi}{\partial x} = \cdots \sin kx = 0 \text{ at } x = 0, \frac{n\pi}{k} = \frac{n\lambda}{2}$$

Therefore, $\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0$. To obtain a standing wave, it is necessary to have perfect reflection at the wall at $x = 0$.

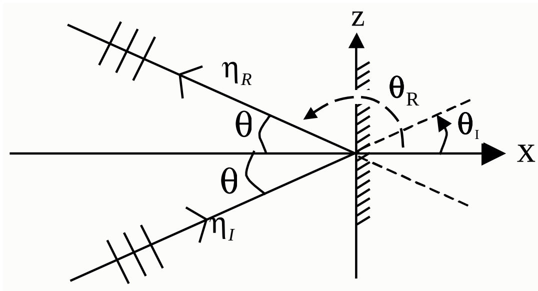
Define the <u>reflection coefficient</u> as $R \equiv \frac{A_R}{A_I}\ (\leq 1)$.



$$A_I = A_R$$
$$R = \frac{A_R}{A_I} = 1$$

3. Oblique Standing Waves.

$$\begin{aligned}\eta_I =& A\cos\left(kx\cos\theta + kz\sin\theta - \omega t\right)\\ \eta_R =& A\cos\left(kx\cos\left(\pi-\theta\right) + kz\sin\left(\pi-\theta\right) - \omega t\right)\end{aligned}$$



$$\theta_R = \pi - \theta_I$$

Note: same A, R = 1.

$$\eta_T = \eta_I + \eta_R = 2A\underbrace{\cos\overbrace{\left(kx\cos\theta\right)}^{k_x x}}_{\text{standing wave in x}}\underbrace{\cos\overbrace{\left(kz\sin\theta - \omega t\right)}^{k_z z - \omega t}}_{\text{propagating wave in z}}$$

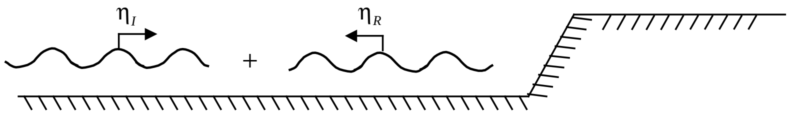
and

$$\lambda_x = \frac{2\pi}{k\cos\theta}; \qquad V_{P_z} = \frac{\omega}{k\sin\theta}; \qquad \lambda_z = \frac{2\pi}{k\sin\theta}$$

Check:

$$\frac{\partial\phi}{\partial \mathrm{x}} \sim \frac{\partial\eta}{\partial \mathrm{x}} \sim \cdots \sin\left(kx\cos\theta\right) = 0 \text{ on } x = 0$$

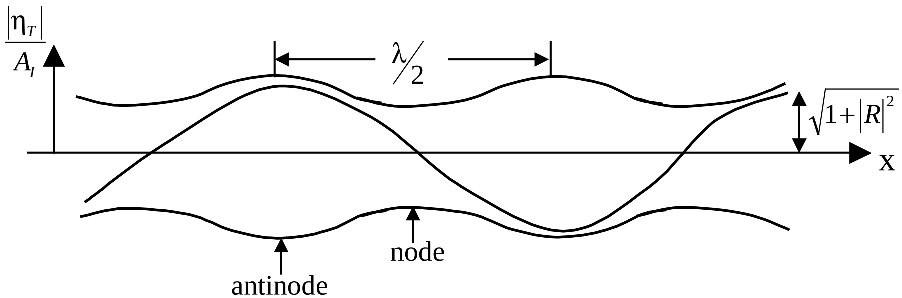
4. Partial Reflection.



$$\eta_I = A_I \cos(kx - \omega t) = A_I Re\left\{e^{i(kx-\omega t)}\right\}$$
$$\eta_R = A_R \cos(kx + \omega t + \delta) = A_I Re\left\{\mathrm{R}\ e^{-i(kx+\omega t)}\right\}$$

R: Complex reflection coefficient

$$\mathrm{R} \ = \ |\mathrm{R}|\, e^{-i\delta}, |\mathrm{R}| = \frac{A_R}{A_I}$$
$$\eta_T = \eta_I + \eta_R = A_I Re\left\{e^{i(kx-\omega t)}\left(1 + \mathrm{R}e^{-2ikx}\right)\right\}$$
$$|\eta_T|^2 = A_I^2\left[1 + |\mathrm{R}|^2 + 2\,|\mathrm{R}|\cos(2kx + \delta)\right]$$



At node,

$$|\eta_T| = |\eta_T|_{\min} = A_I\,(1 - |R|) \ \ at \cos(2kx + \delta) = -1 \text{ or } 2kx + \delta = (2n+1)\,\pi$$

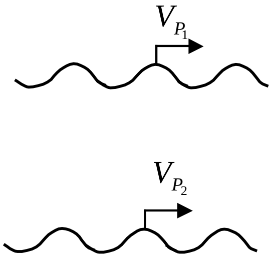
At antinode,

$$|\eta_T| = |\eta_T|_{\max} = A_I\,(1 + |R|) \ \text{ at } \ \cos(2kx + \delta) = 1 \text{ or } 2kx + \delta = 2n\pi$$

$$2kL = 2\pi \text{ so } L = \frac{\lambda}{2}$$

$$|R| = \frac{|\eta_T|_{\max} - |\eta_T|_{\min}}{|\eta_T|_{\max} + |\eta_T|_{\min}} = |R\,(k)|$$

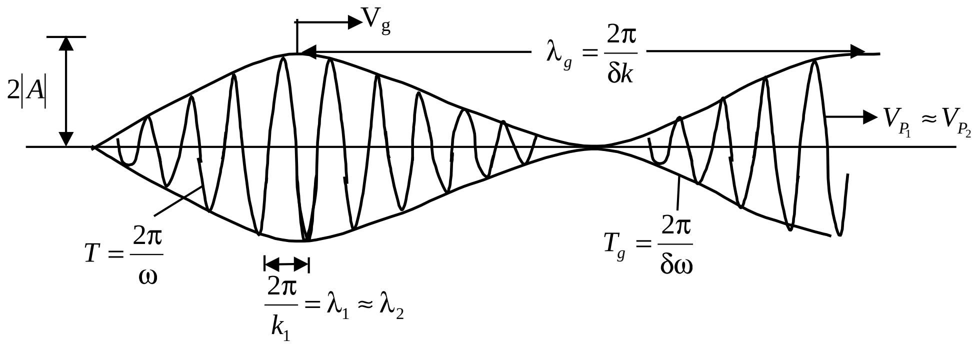
5. <u>Wave Group</u>: 2 waves, same amplitude A and direction, but ω and k very close to each other.



$$\eta_1 = \Re\left(Ae^{i(k_1x-\omega_1t)}\right)$$
$$\eta_2 = \Re\left(Ae^{i(k_2x-\omega_2t)}\right)$$

$$\omega_{1,2} = \omega_{1,2}\left(k_{1,2}\right) \text{ and } V_{P_1} \approx V_{P_2}$$

$$\eta_T = \eta_1 + \eta_2 = \Re\left\{Ae^{i(k_1x-\omega_1t)}\left[1+e^{i(\delta kx-\delta\omega t)}\right]\right\} \text{ with } \delta k = k_2 - k_1 \text{ and } \delta\omega = \omega_2 - \omega_1$$



$$\left.\begin{array}{l} |\eta_T|_{\max} = 2\,|A| \text{ when } \delta kx - \delta\omega t = 2n\pi \\ \\ |\eta_T|_{\min} = 0 \text{ when } \delta kx - \delta\omega t = (2n+1)\,\pi \end{array}\right\} x_g = V_g t,\ \delta k V_g t - (\delta\omega)\,t = 0 \text{ then } V_g = \frac{\delta\omega}{\delta k}$$

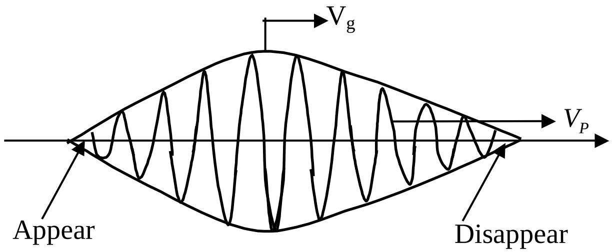
In the limit,

$$\delta k, \delta\omega \to 0, V_g = \left.\frac{d\omega}{dk}\right|_{k_1 \approx k_2 \approx k},$$

and since

$$\omega^2 = gk \tanh kh \Rightarrow$$
$$V_g = \underbrace{\left(\frac{\omega}{k}\right)}_{V_p} \underbrace{\frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)}_{n}$$

$$\left.\begin{array}{l} \text{(a) deep water } kh >> 1 \\ n = \frac{V_g}{V_p} = \frac{1}{2} \\ \text{(b) shallow water } kh << 1 \\ n = \frac{V_g}{V_p} = 1 \text{ (no dispersion)} \\ \text{(c) intermediate depth} \\ \frac{1}{2} < n < 1 \end{array}\right\} V_g \leq V_p$$



<u>Wave Energy</u> - Energy associated with wave motion.

For a single plane progressive wave:

• Potential energy PE (per unit surface area of wave)	• Kinetic energy KE (per unit surface area of wave)
PE without wave $= \int_{-h}^{0} \rho g y dy = -\frac{1}{2}\rho g h^2$ PE with wave $= \int_{-h}^{\eta} \rho g y dy = \frac{1}{2}\rho g \left(\eta^2 - h^2\right)$ $\text{PE}_{\text{wave}} = \frac{1}{2}\rho g \eta^2 = \frac{1}{2}\rho g A^2 \cos^2\left(kx - \omega t\right)$	$\text{KE}_{\text{wave}} = \int_{-h}^{\eta} dy \frac{1}{2}\rho\left(u^2 + v^2\right) = \cdots\cdots$ $= \underbrace{\frac{1}{4}\rho g A^2}_{\substack{\text{for deep water KE} \propto A^2 \\ \text{KE const in x,t}}}$ to leading order
$\overline{\text{PE}}_{\text{wave}} = \frac{1}{4}\rho g A^2$ average over one period or one wavelength	$\overline{\text{KE}}_{\text{wave}} = \frac{1}{4}\rho g A^2$ for <u>any</u> h average over one period or one wavelength

- Wave energy:

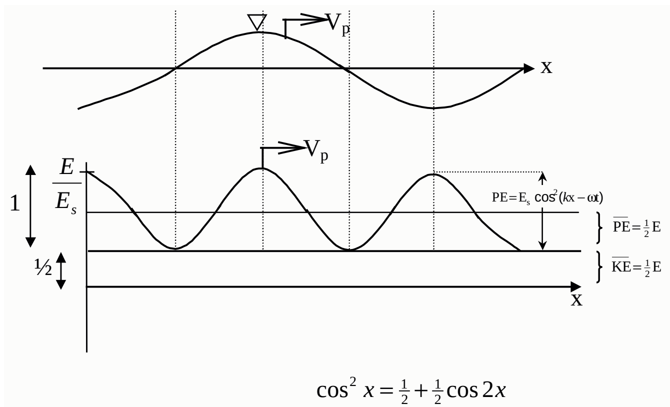
  $$\text{E} \;=\; \text{PE} \;+\; \text{KE} \;= \frac{1}{2}\rho g A^2 \left[\cos^2\left(kx - \omega t\right) + \frac{1}{2}\right]$$

- Average wave energy E (over 1 period or 1 wavelength):

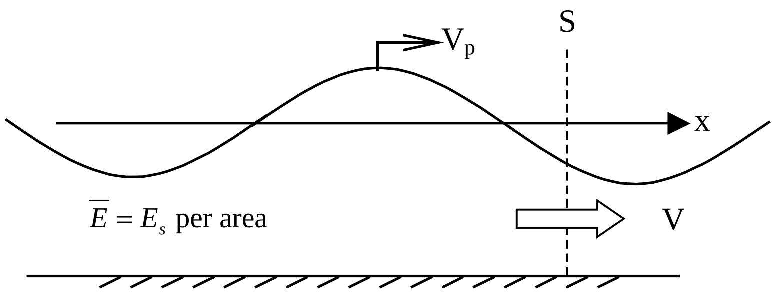
  $$\overline{\text{E}} = \frac{1}{2}\rho g A^2 [\underset{\substack{\uparrow \\ PE}}{\frac{1}{2}} + \underset{\substack{\uparrow \\ KE}}{\frac{1}{2}}] = \frac{1}{2}\rho g A^2 = E_s,$$

  which is the Specific Energy: total average wave energy per unit surface area.

- Linear waves: $\overline{\text{PE}} = \overline{\text{KE}} = \dfrac{1}{2}E_s$ (equipartition).
- Nonlinear waves: $\overline{\text{KE}} > \overline{\text{PE}}$.



Energy Propagation - Group velocity



Consider a fixed control volume V to the right of 'screen' S. Conservation of energy:

$$\underbrace{\frac{dW}{dt}}_{\text{rate of work done on S}} = \underbrace{\frac{dE}{dt}}_{\text{rate of change of energy in V}} = \underbrace{\mathfrak{F}}_{\text{energy flux left to right}}$$

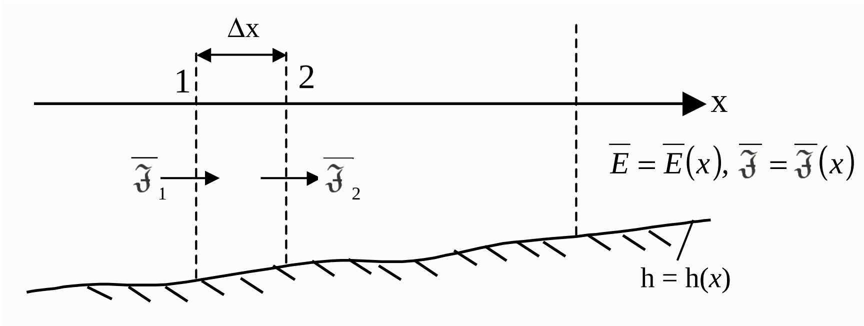
where

$$\mathfrak{F} = \int_{-h}^{\eta} pu\ dy \text{ with } p = -\rho\left(\frac{d\phi}{dt} + gy\right) \text{ and } u = \frac{d\phi}{dx}$$

$$\overline{\mathfrak{F}} = \underbrace{\left(\tfrac{1}{2}\rho g A^2\right)}_{\overline{E}} \underbrace{\underbrace{\left(\frac{\omega}{k}\right)}_{V_p} \underbrace{\left[\tfrac{1}{2}\left(1 + \tfrac{2kh}{\sinh 2kh}\right)\right]}_{n}}_{V_g} = \overline{E}\,(nV_p) = \overline{E}V_g$$

e.g. A = 3m, T = 10 sec $\rightarrow \overline{\mathfrak{F}} = 400KW/m$

Conservation of energy equation



$$\left(\overline{\mathfrak{J}}_1 - \overline{\mathfrak{J}}_2\right)\Delta t = \Delta \overline{E} \Delta x$$

$$\overline{\mathfrak{J}}_2 = \overline{\mathfrak{J}}_1 + \left.\frac{\partial \overline{\mathfrak{J}}}{\partial x}\right|_1 \Delta x + \cdots$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{\mathfrak{J}}}{\partial x} = 0, \text{ but } \overline{\mathfrak{J}} = V_g \overline{E}$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x}\left(V_g \overline{E}\right) = 0$$

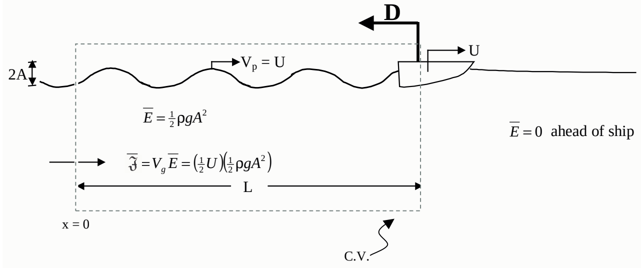
1. $\dfrac{\partial \overline{E}}{\partial t} = 0, V_g \overline{E} =$ constant in x for any $h(x)$.

2. $V_g =$ constant (i.e. constant depth, $\delta k << k$)

$$\left(\frac{\partial}{\partial t} + V_g \frac{\partial}{\partial x}\right)\overline{E} = 0, \text{ so } \overline{E} = \overline{E}\left(x - V_g t\right) \text{ or } A = A\left(x - V_g t\right)$$

i.e. wave packet moves at V_g.

Steady ship waves, wave resistance



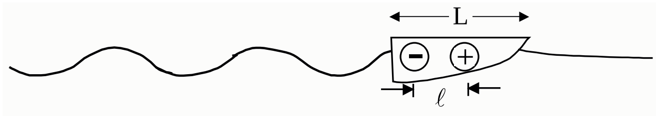
Wave resistance drag on ship D.

$$\text{Rate of work done} = \text{rate of energy increase}$$

$$DU + \overline{\mathfrak{J}} = \frac{d}{dt}\left(\overline{E}L\right) = \overline{E}U$$

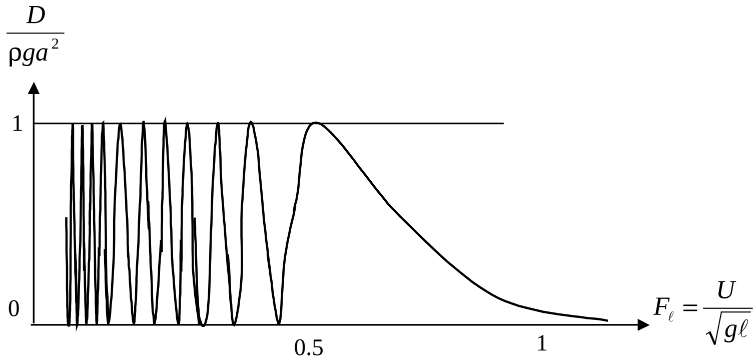
$$\underset{\text{force / length}}{D} = \frac{1}{U}\left(\overline{E}U - \overbrace{\overline{E}U/2}^{\text{deep water}}\right) = \tfrac{1}{2}\overline{E} = \underset{\text{energy / area}}{\tfrac{1}{4}\rho g A^2}$$

Question: Amplitude A = ? (depends on U, geometry). Let $l \equiv$ effective length.



Superimpose a bow wave (η_b) and a stern wave (η_s):

$$\eta_b = a\cos\left(kx\right) \text{ and } \eta_s = -a\cos\left(k\left(x+l\right)\right)$$
$$\eta_T = \eta_b + \eta_s$$
$$A = \left|\eta_T\right|_{\max} = 2a\left|\sin\left(\tfrac{1}{2}kl\right)\right|$$
$$D = \tfrac{1}{4}\rho g A^2 = \rho g a^2 \sin^2\left(\tfrac{1}{2}kl\right),\ V_p = U = \sqrt{\tfrac{g}{k}} \text{ so } k = g/U^2 \text{ deep water}$$
$$D = \rho g a^2 \sin^2\left(\tfrac{1}{2}\tfrac{gl}{U^2}\right)$$



where $\ell \le$ ship length

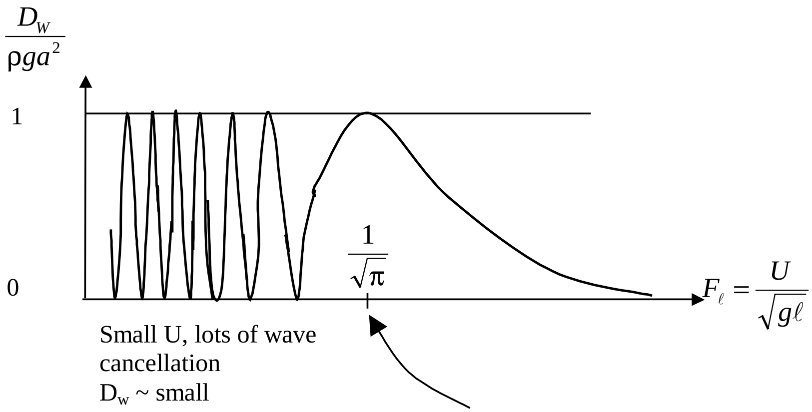
<u>Steady ship waves (deep water)</u>

$$U = \text{ship speed}$$

$$V_p = \sqrt{\frac{g}{k}} = U; \text{ so } k = \frac{g}{U^2} \text{ and } \lambda = 2\pi \frac{U^2}{g}$$

$$L = \text{ship length, } l \sim L$$

$$D_W = \rho g a^2 \sin^2\left(\frac{1}{2}\frac{gl}{U^2}\right) \cong \rho g a^2 \sin^2\left(\frac{1}{\cdots}\right) \cong \rho g a^2 \sin^2\left(\frac{1}{\cdots}\right)$$



$$\frac{1}{2F_\ell^2} = \frac{\pi}{2} \rightarrow F_\ell = \frac{1}{\sqrt{\pi}} \approx 0.56$$

$$\text{"hull speed"} = U_{hull} \approx 0.56\sqrt{g\ell} \cong 0.56\sqrt{gL}$$

$$U_{hull} \propto \sqrt{L}$$