13.021 – Marine Hydrodynamics, Fall 2004
Lecture 3

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1.2 - Stress Tensor

Stress Tensor $\tau_{ij:}$. The stress (force per unit area) at a point in a fluid needs nine components to be completely specified, since each component of the stress must be defined not only by the direction in which it acts but also the orientation of the surface upon which it is acting. The first index specifies the *direction* in which the stress component acts, and the second identifies the orientation of the *surface* upon which it is acting. Therefore, the i^{th} component of the force acting on a surface whose outward normal points in the j^{th} direction is τ_{ij}.

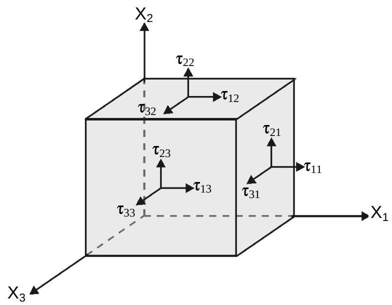

Figure 1: Shear stresses on an infinitesimal cube whose surface are parallel to the coordinate system.

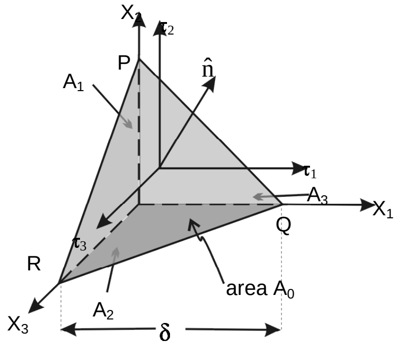

Figure 2: Consider an infinitesimal body at rest with a surface PQR that is not perpendicular to any of the Cartesian axis. The unit normal vector to that surface is $\hat{n} = n_1\hat{x}_1 + n_2\hat{x}_2 + n_3\hat{x}_3$. The area of the surface $= A_0$, and the area of each surface perpendicular to X_i is $A_i = A_0 n_i$, for $i = 1, 2, 3$.

Newton's law: $\sum_{all4faces} F_i =$ (volume force)$_i$ for $i =$ 1, 2, 3

If δ is the typical dimension of the body : surface forces $\sim \delta^2$
: volume forces $\sim \delta^3$

An example of surface forces is the shear force and an example of volumetric forces is the gravity force. At equilibrium, the surface forces and volumetric forces are in balance. As the body gets smaller, the mass of the body goes to zero, which makes the volumetric forces equal to zero and leaving the sum of the surface forces equal zero. So, as $\delta \to 0, \sum_{all4faces} F_i = 0$ for $i = 1, 2, 3$ and $\therefore \tau_i A_0 = \tau_{i1}A_1 + \tau_{i2}A_2 + \tau_{i3}A_3 = \tau_{ij}A_j$. But the area of each surface $\perp$ to X_i is $A_i = A_0 n_i$. Therefore $\tau_i A_0 = \tau_{ij}A_j = \tau_{ij}(A_0 n_j)$, where $\tau_{ij}A_j$ is the $\sum$ notation (represents the sum of all components). Thus $\tau_i = \tau_{ij}n_j$ for i = 1, 2, 3, where τ_i is the component of stress in the i^{th} direction on a surface with a normal $\vec{n}$. We call τ_i the stress vector and we call τ_{ij} the stress matrix or tensor.

Example: Pascal's Law for hydrostatics

In a static fluid, the stress vector cannot be different for different directions of the surface normal since there is no preferred direction in the fluid. Therefore, at any point in the fluid, the stress vector must have the same direction as the normal vector $\vec{n}$ and the same magnitude for all directions of $\vec{n}$.

$$\text{Pascal's Law: for hydrostatics} \quad \tau_{ij} = \overbrace{-(p_i)(\delta_{ij})}^{\text{no summation}}$$

$$\underset{\sim}{\tau} = \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 0 & -p_3 \end{bmatrix}$$

where p_i is the pressure acting perpendicular to the i^{th} surface. If p_o is the pressure acting perpendicular to the surface PQR, then $\tau_i = -n_i p_0$, but $\tau_i = \tau_{ij} n_j = -(p_i)\delta_{ij} n_j = -(p_i)(n_i)$. Therefore $p_o = p_i$, i = 1, 2, 3 and $\vec{n}$ is arbitrary.

Symmetry of the Stress Tensor

To prove the symmetry of the stress tensor we follow the steps:

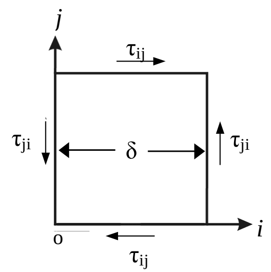

Figure 3: Material element under tangential stress.

1. The $\sum$ of surface forces = body forces + mass× acceleration. Assume no symmetry. Balance of the forces in the i^{th} direction gives:

$$(\delta)(\tau_{ij})_{TOP} - (\delta)(\tau_{ij})_{BOTTOM} = O(\delta^2),$$

   since surface forces are $\sim \delta^2$, where the $O(\delta^2)$ terms include the body forces per unit depth. Then, as $\delta \to 0, (\tau_{ij})_{TOP} = (\tau_{ij})_{BOTTOM}$.

2. The $\sum$ of surface torque = body moment + angular acceleration. Assume no symmetry. The balance moment with respect to o gives:

$$(\tau_{ji}\delta)\delta - (\tau_{ij}\delta)\delta = O(\delta^3),$$

since the body moment is proportional to δ^3. As $\delta \rightarrow 0$, $\tau_{ij} = \tau_{ji}$.

1.3 Mass and Momentum Conservation

Consider a material volume ϑ_m and recall that a material volume is a fixed mass of material. A material volume always encloses the same fluid particles despite a change in size, position, volume or surface area over time.

1.3.1 Mass Conservation

The mass inside the material volume is:

$$M(\vartheta_m) = \iiint\limits_{\vartheta_{m(t)}} \rho d\vartheta$$

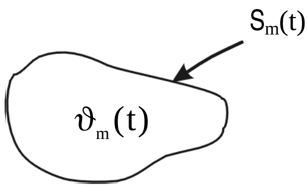

Figure 4: Material volume $\vartheta_m(t)$ with surface $S_m(t)$.

Therefore the time rate of increase of mass inside the material volume is:

$$\boxed{\frac{d}{dt} M(\vartheta_m) = \frac{d}{dt} \iiint\limits_{\vartheta_m(t)} \rho d\vartheta = 0,}$$

which implies conservation of mass for the material volume ϑ_m.

1.3.2 Momentum Conservation

The velocity of fluid inside the material volume in the i^{th} direction is denoted as u_i. Linear momentum of the material volume in the i^{th} direction is

$$\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta$$

Newton's law of motion: The time rate of change of momentum of the fluid in the material control volume must equal the sum of all the forces acting on the fluid in that volume. Thus:

$$\begin{aligned}
\frac{d}{dt}(\text{momentum})_i =& (\text{body force})_i + (\text{surface force})_i \\
\frac{d}{dt}\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta =& \iiint\limits_{\vartheta_m(t)} F_i d\vartheta + \iint\limits_{S_m(t)} \underbrace{\tau_{ij} n_j}_{\tau_i} dS
\end{aligned}$$

Divergence Theorems:

For vectors: $\iiint\limits_{\vartheta} \underbrace{\nabla \cdot \vec{v}}_{\frac{\partial v_j}{\partial x_j}} d\vartheta = \oiint\limits_{S} \underbrace{\vec{v}.\hat{n}}_{v_j n_j} dS$

For tensors: $\iiint\limits_{\vartheta} \frac{\partial \tau_{ij}}{\partial x_j} d\vartheta = \oiint\limits_{S} \tau_{ij} n_j dS$

Thus using divergence theorems:

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta = \iiint\limits_{\vartheta_{m(t)}} \left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) d\vartheta,}$$

which gives the conservation of the momentum for the material volume ϑ_m.

1.4 Kinematic Transport Theorems

Consider a flow through some moving control volume $\vartheta(t)$ during a small time interval Δt. Let $f(\vec{x}, t)$ be any (Eulerian) fluid property per unit volume of fluid (e.g. mass, momentum, etc.). Consider the integral

$$I(t) = \iiint\limits_{\vartheta(t)} f(\vec{x}, t)\, d\vartheta$$

According with the definition of the derivative, we can write

$$\begin{aligned}
\frac{d}{dt} I(t) &= \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iiint\limits_{\vartheta(t+\Delta t)} f(\vec{x}, t+\Delta t) d\vartheta - \iiint\limits_{\vartheta(t)} f(\vec{x}, t) d\vartheta \right\}
\end{aligned}$$

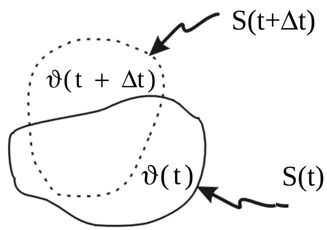

Figure 5: Control volume ϑ and its bounding surface S at instants t and $t + \Delta t$.

Next, we consider the steps

1. Taylor series expansion of f about t.

$$f(\vec{x}, t+\Delta t) = f(\vec{x}, t) + \Delta t \frac{\partial f}{\partial t}(\vec{x}, t) + O((\Delta t)^2)$$

2. $\iiint\limits_{\vartheta(t+\Delta t)} d\vartheta = \iiint\limits_{\vartheta(t)} d\vartheta + \iiint\limits_{\Delta\vartheta} d\vartheta$ where $\iiint\limits_{\Delta\vartheta} d\vartheta = \iint\limits_{S(t)} [U_n(\vec{x}, t)\Delta t]\, dS$ and $U_n(\vec{x}, t)$ is the normal velocity of $S(t)$.

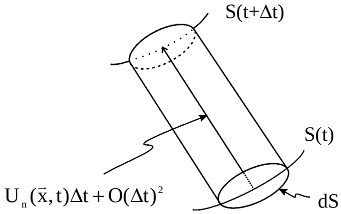

Figure 6: Element of the surface S at instants t and $t+\Delta t$.

So we have

$$\frac{d}{dt}I(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iiint\limits_{\vartheta(t)} d\vartheta f + \Delta t\iiint\limits_{\vartheta(t)} d\vartheta\frac{\partial f}{\partial t} + \Delta t\iint\limits_{S(t)} dSU_n f - \iiint\limits_{\vartheta(t)} d\vartheta f + O(\Delta t)^2\right\}$$

Kinematic Transport Theorem (KTT) $\sim$ Leibnitz rule in 3D

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta(t)} f(\vec{x},t)d\vartheta = \iiint\limits_{\vartheta(t)}\frac{\partial f(\vec{x},t)}{\partial t}d\vartheta + \iint\limits_{S(t)} f(\vec{x},t)U_n(\vec{x},t)dS}$$

If the control volume is a material volume: $\vartheta(t) = \vartheta_m(t)$ and $\mathrm{U}_n = \vec{v}\cdot\hat{n}$, where $\vec{v}$ is the fluid particle velocity. Then the Kinematic Transport theorem (KTT) assume the form

$$\frac{d}{dt}\iiint\limits_{\vartheta_m(t)} f(\vec{x},t)d\vartheta = \iiint\limits_{\vartheta_m(t)}\frac{\partial f(\vec{x},t)}{\partial t}d\vartheta + \iint\limits_{S_m(t)}\underbrace{f(\vec{x},t)(\vec{v}\cdot\hat{n})}_{fv_in_i(\text{Einstein Notation})}dS$$

Using the divergence theorem:

$$\iiint\limits_{\vartheta}\underbrace{\nabla\cdot\vec{\alpha}}_{\frac{\partial}{\partial x_i}\alpha_i}d\vartheta = \oiint\limits_{S}\underbrace{\vec{\alpha}\cdot\hat{n}}_{\alpha_i n_i}dS \tag{1}$$

1^{st} Kinematic Transport Theorem (KTT)

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_m(t)} f\left(\vec{x},t\right)d\vartheta = \iiint\limits_{\vartheta_m(t)}\left[\frac{\partial f(\vec{x},t)}{\partial t} + \underbrace{\nabla\cdot\left(f\vec{v}\right)}_{\frac{\partial}{\partial x_i}(fv_i)}\right]d\vartheta,}$$

where f is a fluid property per unit volume.

1.5 Continuity Equation

Let the fluid property per unit volume be mass per unit volume ($f = \rho$)

$$0 \underset{\substack{\uparrow \\ \text{conservation} \\ \text{of mass}}}{=} \frac{d}{dt}\iiint\limits_{\vartheta_m(t)} \rho d\vartheta \underset{\substack{\uparrow \\ 1^{st}\text{KTT}}}{=} \iiint\limits_{\vartheta_m(t)}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right]d\vartheta$$

since ϑ_m is arbitrary, so the integrand $\equiv 0$ everywhere. Therefore, the differential form of conservation of mass i.e. Continuity equation follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$
$$\underbrace{\frac{\partial\rho}{\partial t} + [\vec{v}\cdot\nabla\rho}_{\frac{D\rho}{Dt}} + \rho\nabla\cdot\vec{v}] = 0$$

Therefore,

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0}$$

In general, $\rho = \rho(p, T, \ldots)$. We consider the special case of <u>incompressible flow</u> (Note, the density of the entire flow is not constant when we have more than one fluid, like water and oil, as illustrated in the picture above).

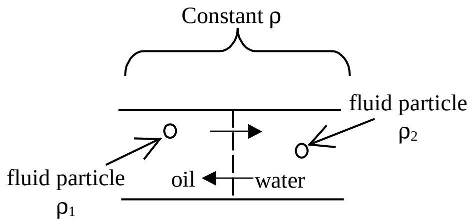

Figure 7: Interface of two fluids (oil-water)

Therefore, for an incompressible flow:

$$\boxed{\frac{D\rho}{Dt} = 0}$$

Then $\underbrace{\nabla \cdot \vec{v} \text{ or } \frac{\partial v_i}{\partial x_i}}_{\text{rate of volume dilatation}} = 0$, which is the Continuity equation for incompressible fluid.

1.6 Euler's Equation (differential form of conservation of momentum)

2^{nd} Kinematic Transport Theorem ($= 1^{st}$ KTT + continuity equation). If G = fluid property per unit mass, then ρG = fluid property per unit volume

$$\begin{aligned} \frac{d}{dt}\iiint\limits_{\vartheta_m(t)} \rho G d\vartheta &= \iiint\limits_{\vartheta_m(t)} \left[\frac{\partial}{\partial t}(\rho G) + \nabla \cdot (\rho G \vec{v})\right] d\vartheta \\ &= \iiint\limits_{\vartheta_m(t)} \left[G \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v}\right)}_{0 \text{ from mass conservation}} + \rho \underbrace{\left(\frac{\partial G}{\partial t} + \vec{v} \cdot \nabla G\right)}_{\frac{DG}{Dt}} \right] d\vartheta, \end{aligned}$$

and the 2^{nd} Kinematic Transport Theorem (KTT) follows:

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_m} \rho G d\vartheta = \iiint\limits_{\vartheta_m} \rho \frac{DG}{Dt} d\vartheta}$$

Application:

We consider G as the i^{th} momentum per unit mass (v_i). Then,

$$\iiint\limits_{\vartheta_m(t)} \left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) d\vartheta \underset{\substack{\uparrow \\ \text{conservation} \\ \text{of momentum}}}{=} \frac{d}{dt}\iiint\limits_{\vartheta_m(t)} \rho v_i d\vartheta \underset{\substack{\uparrow \\ 2^{nd}\text{KTT}}}{=} \iiint\limits_{\vartheta_m(t)} \rho\frac{Dv_i}{Dt} d\vartheta$$

But $\vartheta_m(t)$ is an arbitrary material volume, therefore the integral identity gives **Euler's equation**

$$\boxed{\rho\frac{Dv_i}{Dt} \equiv \rho\left(\frac{\partial v_i}{\partial t} + \underbrace{\vec{v}\cdot\nabla v_i}_{v_j\frac{\partial v_i}{\partial x_j}}\right) = F_i + \frac{\partial \tau_{ij}}{\partial x_j},}$$

and its Vector Tensor Form

$$\boxed{\rho\frac{D\vec{v}}{Dt} \equiv \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \vec{F} + \nabla\cdot\underset{\sim}{\tau}}$$