13.021 - Marine Hydrodynamics
Lecture 4

Introduction

Governing Equations so far:

Knowns	Number of Equations		Number of Unknowns	
ρ	Continuity(conservation of mass)	1	v_i	3
F_i	Euler (conservation of momentum)	3	τ_{ij}	6 (3 of 9 eliminated by symmetry)
		4		9

Therefore, some constitutive relationships are needed to relate v_i to τ_{ij}.

1.7 Newtonian Fluid

1. Consider a fluid at rest ($v_i \equiv 0$). Then according to Pascal's Law:

$$\tau_{ij} = -p_s \delta_{ij} \text{ (Pascal's Law)}$$

$$\underset{\sim}{\tau} = \begin{bmatrix} -p_s & 0 & 0 \\ 0 & -p_s & 0 \\ 0 & 0 & -p_s \end{bmatrix}$$

   where p_s is the hydrostatic pressure and δ_{ij} is the Kroenecker delta function, equal to 1 if $i = j$ and 0 if $i \neq j$.

2. Consider a fluid in motion. The fluid stress is defined as:

$$\tau_{ij} = -p\delta_{ij} + \hat{\tau}_{ij}$$

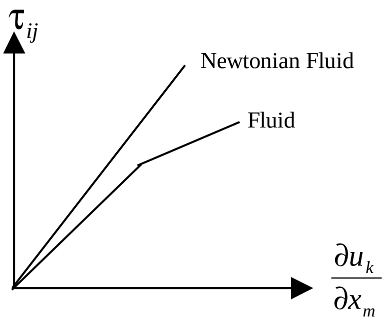
where p is the thermodynamic pressure and $\hat{\tau}_{ij}$ is the dynamic stress, which is related to the velocities empirically.

Experiments with a wide class of fluids, "*Newtonian*" fluids, obtain that:

$$\hat{\tau}_{ij} \approx \text{ linear function of the } \overbrace{\frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x}\right) = \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial X}{\partial t}\right)}_{u}}^{\underline{\text{rate of strain}}} \equiv \overbrace{\frac{\partial u_k}{\partial x_m}}^{\underline{\text{velocity gradient}}}$$

$$\text{i.e.} \quad \hat{\tau}_{ij} \approx \underbrace{\alpha_{ijkm}}_{\substack{3^4=81 \\ \text{empirical coefficients} \\ \text{(constants for Newtonian fluids)}}} \frac{\partial u_k}{\partial x_m} \qquad i,j,k,m = 1,2,3$$

Note that the shear stress is proportional to the rate of strain.



For isotropic fluids, this reduces to:

$$\hat{\tau_{ij}} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\underbrace{\left(\frac{\partial u_l}{\partial x_l}\right)}_{\nabla\cdot\vec{v}},$$

where the fluid properties are:

- μ - (coefficient of) dynamic viscosity.
- λ - bulk elasticity, 'second' coefficient of viscosity

For incompressible flow, $\frac{\partial u_l}{\partial x_l} = 0$. Therefore, for an incompressible, isotropic, Newtonian fluid the viscous stress $\hat{\tau}_{ij}$ is given as

$$\hat{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

1.8 Navier-Stokes equations

Equations	Number of Equations	Unknowns	Number of Unknowns
Euler	3	u_i	3
continuity	1	p	1
constitutive (Newtonian)	6 (symmetry)	τ_{ij}	6
	10		10

closure

Substitute the equation for the stress tensor

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

for a Newtonian fluid into Euler's equation:

$$\rho \frac{Du_i}{Dt} = F_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

where

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \underbrace{\frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i} \underbrace{\frac{\partial u_j}{\partial x_j}}_{0}}$$

and $\frac{\partial u_j}{\partial x_j} = 0$ due to continuity. Finally,

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho}F_i \quad \text{Tensor form}$$

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} + \frac{1}{\rho}\vec{F} \qquad \text{Vector form}$$

where $\nu \equiv \frac{\mu}{\rho}$ denoted as the **Kinematic viscosity** [L^2/T].

- Navier-Stokes equations for incompressible, Newtonian fluids

	Number of Equations	Number of Unknowns	
continuity	1	p	1
Navier-Stokes	3 (symmetry)	u_i	3
	4		4

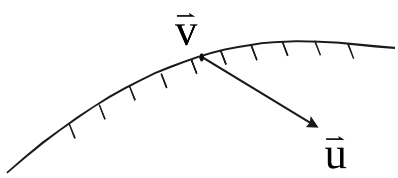
1.9 Boundary Conditions

1. **Kinematic Boundary Conditions**: Specifies kinematics (position, velocity, ...) On a solid boundary, velocity of the fluid = velocity of the body. i.e. velocity continuity.

$$\vec{v} = \vec{u} \quad \text{"no-slip" boundary condition}$$

where $\vec{v}$ is the fluid velocity at the body and $\vec{u}$ is the body surface velocity

- $\vec{v}\cdot\hat{n} = \vec{u}\cdot\hat{n}$ no flux – continuous flow
- $\vec{v}\cdot\hat{t} = \vec{u}\cdot\hat{t}$ no slip – finite shear stress

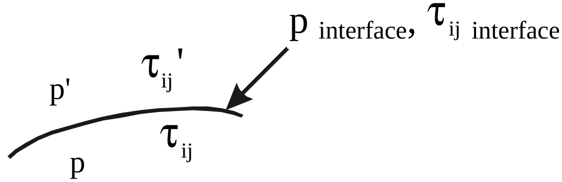


2. **Dynamic Boundary Conditions**: Specifies dynamics (pressure, sheer stress, ...)

Stress continuity:

$$p = p' + p_{\text{interface}}$$
$$\tau_{ij} = \tau'_{ij} + \tau_{ij\ \text{interface}}$$

The most common example of interfacial stress is surface tension.



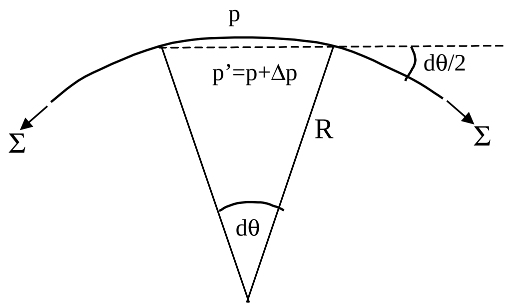
Surface Tension

- Notation: Σ [Tension force / Length] $\equiv$ [Surface energy / Area].
- Surface tension is due to the inter molecular forces attraction forces in the fluid.
- At the interface of two fluids, surface tension implies in a pressure jump across the interface. Σ gives rise to Δp across an interface.
- For a water/air interface: $\Sigma = 0.07$ N/m. This is a function of temperature, impurities etc...
- 2D Example:

$$\underbrace{\cos\frac{d\theta}{2}}_{\approx 1} \cdot \Delta p \cdot R d\theta = 2\Sigma \underbrace{\sin\frac{d\theta}{2}}_{\approx \frac{d\theta}{2}} \approx 2\Sigma\tfrac{d\theta}{2}$$

$$\therefore \Delta p = \tfrac{\Sigma}{R}$$

Higher curvature implies in higher pressure jump at the interface.



- 3D Example: Compound curvature

$$\Delta p = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \Sigma$$

where R_1 and R_2 are the principle radii of curvature.

1.10 Body Forces – Gravity

- Conservative force:

$$\vec{F} = -\nabla \varphi \quad \text{for some } \varphi,$$

where φ is the force potential.

$$\oint \vec{F} \cdot d\vec{x} = 0 \text{ or } \int_1^2 \vec{F} \cdot d\vec{x} = -\int_1^2 \nabla \varphi \cdot d\vec{x} = \varphi(\vec{x}_1) - \varphi(\vec{x}_2)$$

- A special case of a conservative force is gravity.

$$\vec{F} = -\rho g \hat{k},$$

with gravitational potential:

$$\varphi = \rho g z \rightarrow \vec{F} = -\nabla \varphi = \nabla(-\rho g z) = -\rho g \hat{k},$$

where $-\rho g z$ is the hydrostatic pressure $p_s = -\rho g z = -\varphi$.

Navier-Stokes equation:

$$
\begin{aligned}
\rho \frac{D\vec{v}}{Dt} &= -\nabla p + \vec{F} + \rho\nu\nabla^2\vec{v} \\
&= -\nabla p - \nabla \rho g z + \rho\nu\nabla^2\vec{v} \\
&= -\nabla\left(p + \rho g z\right) + \rho\nu\nabla^2\vec{v},
\end{aligned}
$$

but $p - p_s = p_d$ and $\mathrm{p}_s = -\rho g z$, where p is the total pressure and p_d is the dynamic pressure. Therefore,

$$
\rho \frac{D\vec{v}}{Dt} = -\nabla p_d + \rho\nu\nabla^2\vec{v}
$$

- Presence of gravity body force is equivalent to replacing total pressure p by dynamic pressure p_d in the Navier-Stokes(N-S) equation.
- Solve the N-S equation with p_d, then calculate $p = p_d + p_s = p_d - \rho g z$.