13.021 - Marine Hydrodynamics
Lecture 6

2.2 Similarity Parameters (from governing equations)

Non-dimensionalize and normalize basic equations by scaling:

Identify characteristic scales for the problem

velocity	U	$\vec{v} = U\vec{v}*$
length	L	$\vec{x} = L\vec{x}*$
time	T	$t = Tt*$
pressure	p_o- p_v	$p = (p_o - p_v)p*$

All ()* quantities are dimensionless and normalized (i.e. O(1)), e.g. $\frac{\partial \vec{v}*}{\partial x*} = O(1)$.
Apply to governing equations: (also internal constitution, boundary conditions)

- Continuity (incompressible flow):

$$\nabla \cdot \vec{v} = \frac{U}{L}\nabla^* \cdot \vec{v}^* = 0, \quad \nabla^* \cdot \vec{v}^* = 0$$

- Navier-Stokes:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \vec{v} = -\frac{1}{\rho}\nabla p + \upsilon \nabla^2 \vec{v} - g\hat{j}$$

$$\frac{U}{T}\frac{\partial \vec{v}^*}{\partial t^*} + \frac{U^2}{L}\left(\vec{v}^* \cdot \nabla^*\right)\vec{v}^* = -\frac{p_o - p_v}{\rho L}\nabla^* p^* + \frac{\upsilon U}{L^2}\nabla^{*2}\vec{v}^* - g\hat{j}$$

divide through by $\frac{U^2}{L}$(order of magnitude of the convective inertia term)

$$\widetilde{\frac{L}{UT}}\left(\frac{\partial \vec{v}}{\partial t}\right)^* + \left(\left(\vec{v} \cdot \nabla\right)\vec{v}\right)^* = -\widetilde{\frac{p_o - p_v}{\rho U^2}}\left(\nabla p\right)^* + \widetilde{\frac{\nu}{UL}}\left(\nabla^2 \vec{v}\right)^* - \widetilde{\frac{gL}{U^2}}\hat{j}$$

Since all $()^*$ terms are O(1), the coefficients ($\widetilde{\quad\quad}$) measure the relative importance of each term (as compared to the convective inertia term):

- $\frac{L}{UT} = S =$ Strouhal number $\sim \dfrac{\text{Eulerian inertia } \frac{\partial \vec{v}}{\partial t}}{\text{convective inertia } (\vec{v}\cdot\nabla)\vec{v}}$
  is a measure of transient behavior. For example e.g. if $T >> \frac{L}{U}, S << 1$, ignore $\frac{\partial \vec{v}}{\partial t} \longrightarrow$ assume steady-state.

- $\frac{p_o - p_v}{\frac{1}{2}\rho U^2} = \sigma =$ cavitation number (measures likelihood of cavitation)

  If $\sigma \quad >> \quad 1$, no cavitation. Alternatively, when cavitation is not a concern $p = p_o p^*$.

- $\frac{p_o}{\frac{1}{2}\rho U^2} = E_u =$ Euler number $\sim \dfrac{\text{pressure force}}{\text{inertia force}}$.

- $\frac{UL}{\nu} = R_e =$ Reynold's number $\sim \dfrac{\text{inertia force}}{\text{viscous force}}$

  If $R_e >> 1$, ignore viscosity.

- $\sqrt{\frac{U^2}{gL}} = \frac{U}{\sqrt{gL}} = F_r =$ Froude number $\sim \left(\dfrac{\text{inertia force}}{\text{gravity force}}\right)^{\frac{1}{2}}$

  – Kinematic boundary conditions: $\vec{v} = \vec{U}_b \longrightarrow \vec{v}^* = \vec{U}_b^*$
  – Dynamic boundary conditions:

$$p = p_a + \Delta p \;\text{ where }\; \Delta p = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sum$$

$$p^* = p_a^* + \frac{\sum}{(p_o - p_v)\,L}\left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right)$$

$$\text{where } \frac{\sum}{(p_o - p_v)\,L} = \frac{2}{\sigma}\frac{\sum/\rho}{U^2 L}$$

- $\frac{U^2 L}{\sum/\rho} = W_e =$ Weber number $\sim \dfrac{\text{inertial forces}}{\text{surface tension forces}}$
  note: $L >> R_o$ usually

Alternatively, using physical arguments: forces acting on a fluid particle

1. inertial forces $\sim$ mass $\times$ acceleration $\sim (\rho L^3)\left(\frac{U^2}{L}\right) = \rho U^2 L^2$

2. viscous forces $\sim \underbrace{\mu \dfrac{\partial u}{\partial y}}_{\text{shear stress}} \times$ area $\sim \left(\mu \frac{U}{L}\right)(L^2) = \mu U L$

3. gravitational forces $\sim$ mass $\times$ gravity $\sim (\rho L^3) g$

4. pressure forces $\sim (p_o - p_v) L^2$

For similar streamlines, particles must be acted on by forces whose resultant is in the same direction at geosimilar points. Therefore, forces must be in the same ratios:

$$\frac{inertia}{viscous} \sim \frac{\rho U^2 L^2}{\mu U L} = \frac{UL}{\upsilon} = R_e$$

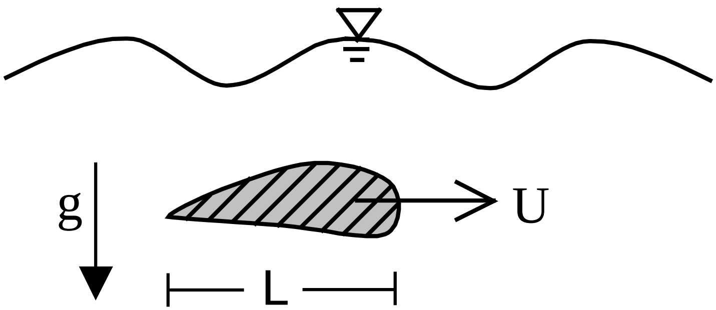
$$\left(\frac{inertia}{gravity}\right)^{1/2} \sim \left(\frac{\rho U^2 L^2}{\rho g L^3}\right)^{1/2} = \frac{U}{\sqrt{gL}} = F_r$$

$$\left(\frac{\frac{1}{2} inertia}{pressure}\right)^{-1} \sim \frac{(p_o - p_v) L^2}{\frac{1}{2}\rho U^2 L^2} = \frac{p_o - p_v}{\frac{1}{2}\rho U^2} = \sigma$$

Importance of Various Parameters

- Govern flow similitude of different systems.
- Provide guidance and approximate the complex physical problem.

e.g.



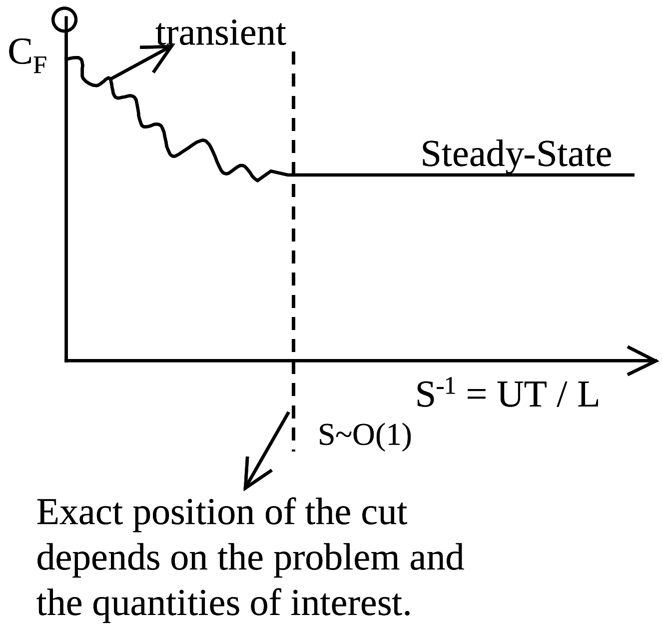
Parameters:

$$S = \frac{L}{UT}, \sigma = \frac{P_o - P_v}{\frac{1}{2}\rho U^2}, W_e = \frac{U^2 L}{\frac{\Sigma}{\rho}}, F_r = \frac{U}{\sqrt{gL}}, R_e = \frac{UL}{\nu}$$

Force coefficient on the foil:

$$C_F = \frac{F}{\frac{1}{2}\rho U^2 L^2} = C_F\left(S, \sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}\right)$$

1. $S = L/UT$, change S with σ, W_e, F_r, R_e fixed.



For $S << 1$, assume steady-state: $\frac{\partial}{\partial t} = 0$
For $S >> 1$, unsteady effect is dominant. For example:

$$\left\{ \begin{array}{c} L \approx 10\text{m} \\ U \approx 10\text{m/s} \end{array} \right. \Rightarrow T \approx 1 \text{ sec} \quad \text{gives } S \approx 1, \therefore \text{ for } T >> 1\text{sec assume steady state since } S << 1$$

So, for steady-state problem:

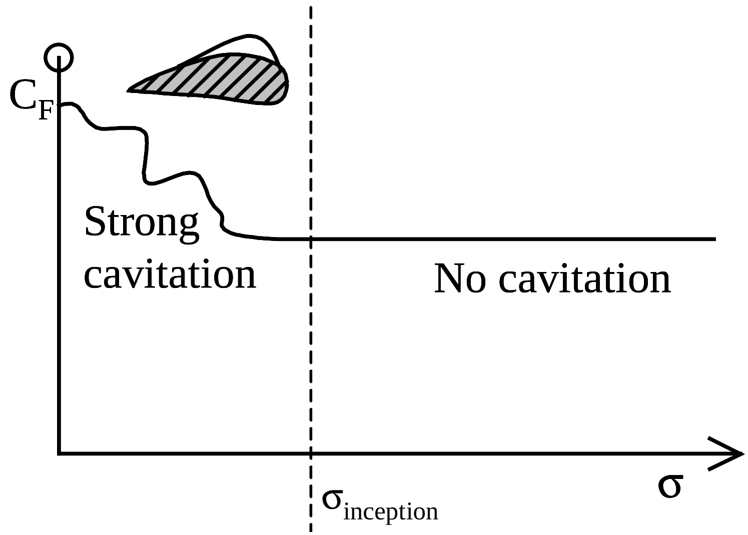
$$C_F = C_F \left(\sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}\right)$$

2. $\sigma = \frac{P_o - P_v}{\frac{1}{2}\rho U^2}$ (fixed R_e, F_r and W_e).

P_v: Vapor pressure
$P_o \leq P_v$: State of fluid changes from liquid to gas ← CAVITATION

Mechanism: $P_o < P_v$ → Fluids cannot withstand tensions, the state of fluids changes.

Consequence: (1) Unsteady → Vibration of the structures, which may lead to fatigue
(2) Unstable → Sudden cavity collapses → huge force acting on the structure surface → surface erosion.



For $\sigma << 1$, there is cavitation, and for $\sigma >> 1$, there is *no* cavitation. For example:

$$\left\{\begin{array}{c} p_0 \approx 10^5 \mathrm{N}/m^2 \\ p_v \approx 2\times 10^3 \mathrm{n}/m^2 \\ \rho \approx 10^3 \mathrm{kg}/m^3 \\ L \approx 100\mathrm{m} \\ U \approx 10\mathrm{m/s} \end{array}\right. \quad \Rightarrow \sigma = 2. \text{ To have cavitation we need large } U \text{ or } p_o \sim p_v$$

Note: p_v is the pressure at which the water boils.

For steady non-cavitation flow ($\sigma >> 1$)

$$C_F = C_F\left(W_e^{-1}, F_r, R_e^{-1}\right)$$

3. $W = \frac{U^2 L}{\frac{\Sigma}{\rho}}$ (fixed R_e and F_r). For example, if $U = 1m/s, \sum = 0.07N/m$(water-air 20^{o}C), ρ=10^{3}kg/m^3 and $L = 100$ m, we end up with $W_e \approx 10^8$. If we want $W_e \approx 1$, we need $L \approx 10^{-4}$ m. Then, for $L >> 10^{-4}$ m, $W_e \to \infty$ and $W_e^{-1} \to 0$, so neglect surface tension effect.

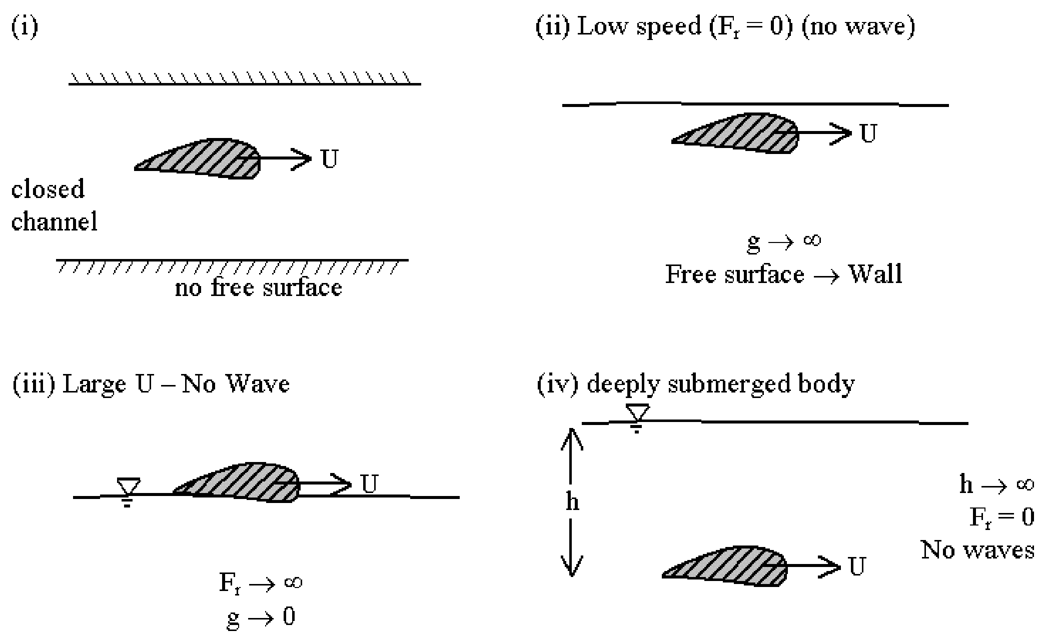
For steady, non-cavitation, non-surface tension effect,

$$C_F = C_F\left(F_r, R_e^{-1}\right)$$

4. $F_r = \frac{U}{\sqrt{gL}}$, which measures the effect of gravity.

   For problems without dynamic boundary conditions (i.e. if free surface is absent) or if the free-surface is far away or not displaced, gravity effects are irrelevant and F_r is not important $\rightarrow \quad F^* = C_F\left(R_e^{-1}\right)$

   e.g.



In general $C_F = C_F\left(F_r, R_e^{-1}\right) = C_1\left(F_r\right) + C_2\left(R_e^{-1}\right)$ $\leftarrow$ Froude's Hypothesis

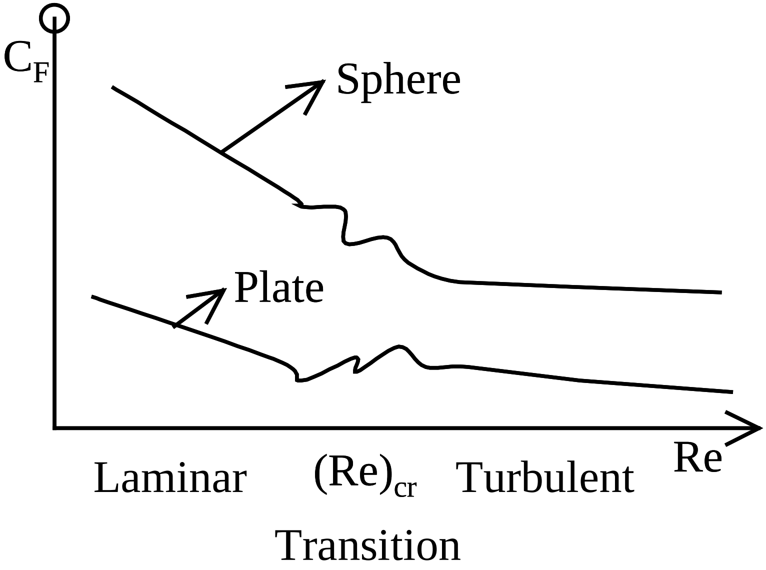
Dynamic similarity requires:

$$(R_e)_1 = (R_e)_2,$$
$$(F_r)_1 = (F_r)_2.$$

For two geometrically similar systems $\rightarrow U_1 = U_2,\ L_1 = L_2$ for the same ν and g.

5. $R_e = UL/\nu$.

   For steady, no σ, no W_e, no gravity effects, $C_F = C_F\left(R_e^{-1}\right)$



$$R_e << 1, \text{ Stokes flow (creeping flow)}$$
$$R_e < (R_e)_{cr}, \text{ Laminar flow}$$
$$R_e > (R_e)_{cr}, \text{ Turbulent flow}$$
$$R_e \to \infty, \text{ Ideal flow}$$

For example:

$$\left\{ \begin{array}{c} U = 10m/s \\ L = 10m \\ \nu = 10^{-6} m^2/sec \end{array} \right. \Rightarrow R_e = 10^8 \text{ or } \; R_e^{-1} = 10^{-8}$$

For steady, no σ, no W_e, no gravity effect and ideal fluid:

$$C_F = C_F\,(0,0,0,0,0) = \text{constant} = 0$$

→D'Alembert's Paradox: No drag force on moving body.