13.021 - Marine Hydrodynamics
Lecture 9

Vorticity Equation

Return to viscous incompressible flow.

N-S equation: $\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla\left(\frac{p}{\rho} + gy\right) + \nu \nabla^2 \vec{v}$

$\nabla \times () \longrightarrow \frac{\partial \vec{\omega}}{\partial t} + \nabla \times \left(\vec{v} \cdot \nabla \vec{v}\right) = \nu \nabla^2 \vec{\omega}$ since $\nabla \times \nabla \phi = 0$ for any ϕ (conservative forces)

Now:

$$
\begin{aligned}
(\vec{v} \cdot \nabla)\vec{v} &= \frac{1}{2}\nabla\left(\vec{v} \cdot \vec{v}\right) - \vec{v} \times \left(\nabla \times \vec{v}\right) \\
&= \nabla\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} \text{ where } v^2 \equiv \left|\vec{v}\right|^2 = \vec{v} \cdot \vec{v} \\
\nabla \times \left(\vec{v} \cdot \nabla\right)\vec{v} &= \nabla \times \nabla\left(\frac{v^2}{2}\right) - \nabla \times \left(\vec{v} \times \vec{\omega}\right) = \nabla \times \left(\vec{\omega} \times \vec{v}\right) \\
&= \left(\vec{v} \cdot \nabla\right)\vec{\omega} - \left(\vec{\omega} \cdot \nabla\right)\vec{v} + \underbrace{\vec{\omega}\left(\nabla \cdot \vec{v}\right)}_{\substack{=0 \\ \text{incompressible} \\ \text{fluid}}} + \underbrace{\vec{v}\left(\nabla \cdot \vec{\omega}\right)}_{\substack{=0 \text{ since} \\ \nabla \cdot (\nabla \times \vec{v}) = 0}}
\end{aligned}
$$

Therefore,

$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \vec{\omega} = \left(\vec{\omega} \cdot \nabla\right) \vec{v} + \nu \nabla^2 \vec{\omega}$$

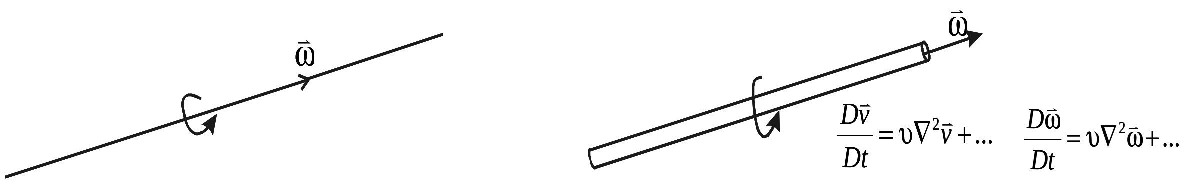
or

$$\frac{D\vec{\omega}}{Dt} = \left(\vec{\omega} \cdot \nabla\right) \vec{v} + \underbrace{\nu \nabla^2 \vec{\omega}}_{\text{diffusion}}$$

- Kelvin's Theorem revisited.

  If $\nu \equiv 0$, then $\frac{D\vec{\omega}}{Dt} = \left(\vec{\omega} \cdot \nabla\right) \vec{v}$, so if $\vec{\omega} \equiv 0$ everywhere at one time, $\vec{\omega} \equiv 0$ always.

- ν can be thought of as diffusivity of (momentum) and vorticity, i.e., $\vec{\omega}$ once generated (on boundaries only) will spread/diffuse in space if ν is present.



- Diffusion of vorticity is analogous to the heat equation: $\frac{\partial T}{\partial t} = K\nabla^2 T$, where K is the heat diffusivity

  Also since $\nu \sim$ 1 or 2 mm^2/s, in 1 second, diffusion distance $\sim O\left(\sqrt{\nu t}\right) \sim O\left(mm\right)$, whereas diffusion time $\sim O\left(L^2/\nu\right)$. So for a diffusion distance of L = 1cm, the necessary diffusion time needed is O(10)sec.

- For 2D, $\vec{v} = (u, v, 0)$ and $\frac{\partial}{\partial z} \equiv 0$. So, $\vec{\omega} = \nabla \times \vec{v}$ is $\perp$ to $\vec{v}$ (parallel to z-axis).

  Then,

$$(\vec{\omega} \cdot \nabla)\, \vec{v} = \left(\underbrace{\omega_x}_{0} \frac{\partial}{\partial x} + \underbrace{\omega_y}_{0} \frac{\partial}{\partial y} + \omega_z \underbrace{\frac{\partial}{\partial z}}_{0} \right) \vec{v} \equiv 0,$$

  so in 2D we have

$$\frac{D\vec{\omega}}{Dt} = \nu \nabla^2 \vec{\omega}.$$

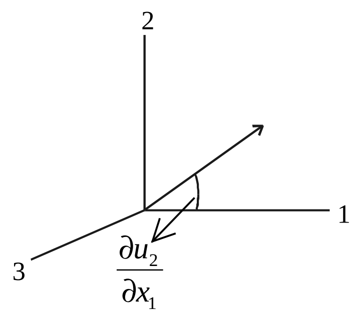
  If $\nu = 0$, $\frac{D\vec{\omega}}{Dt} = 0$, i.e. in 2D, following a particle, the angular velocity is conserved. **Reason**: in 2D, the length of a vortex tube cannot change due to continuity.

- For 3D,

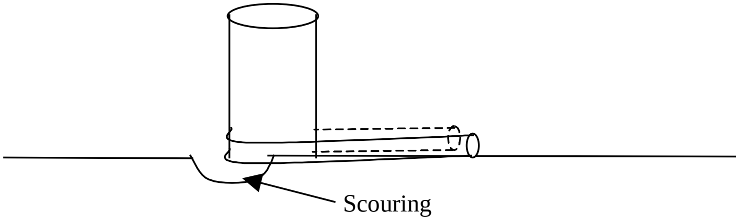
$$\frac{D\omega_i}{Dt} = \underbrace{\omega_j \frac{\partial v_i}{\partial x_j}}_{\text{vortex turning and stretching}} + \underbrace{\nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j}}_{diffusion}$$

  e.g.

$$\frac{D\omega_2}{Dt} = \underbrace{\omega_1 \frac{\partial u_2}{\partial x_1}}_{\text{vortex turning}} + \underbrace{\omega_2 \frac{\partial u_2}{\partial x_2}}_{\text{vortex stretching}} + \underbrace{\omega_3 \frac{\partial u_2}{\partial x_3}}_{\text{vortex turning}} + \text{diffusion}$$



Example: Pile on a River



What really happens as length of the vortex tube L increases?

IFCF is no longer a valid assumption.

Why?

Ideal flow assumption implies that the inertia forces are much larger than the viscous effects (Reynolds number).

$R_e \sim \frac{UL}{\nu}$

Length increases $\Rightarrow$ diameter becomes really small $\Rightarrow$ R_e is not that big after all.

Therefore IFCF is no longer valid.

3.3 Potential Flow - ideal (inviscid and incompressible) and irrotational flow

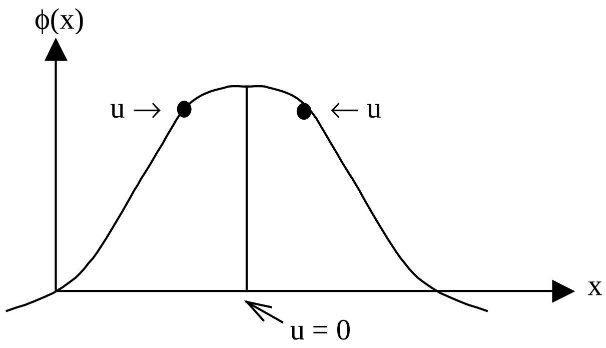
If $\vec{\omega} \equiv 0$ at some time t, then $\vec{\omega} \equiv 0$ always for ideal flow under conservative body forces by Kelvin's theorem.

Given a vector field $\vec{v}$ for which $\vec{\omega} = \nabla \times \vec{v} \equiv 0$, then there exists a potential function (scalar) - the velocity potential - denoted as ϕ, for which

$$\vec{v} = \nabla \phi$$

Note that $\vec{\omega} = \nabla \times \vec{v} = \nabla \times \nabla \phi \equiv 0$ for any ϕ, so irrotational flow guaranteed automatically. At a point $\vec{x}$ and time t, the velocity vector $\vec{v}(\vec{x}, t)$ in cartesian coordinates in terms of the potential function $\phi(\vec{x}, t)$ is given by

$$\vec{v}\left(\vec{x}, t\right) = \nabla \phi \left(\vec{x}, t\right) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$



The velocity vector $\vec{v}$ is the gradient of the potential function ϕ, so it always points towards higher values of the potential function.

Governing Equations:

Continuity:

$$\nabla \cdot \vec{v} = 0 = \nabla \cdot \nabla\phi \Rightarrow \nabla^2\phi = 0$$

Number of unknowns $\rightarrow \quad \phi$

Number of equations $\rightarrow \quad \nabla^2\phi = 0$

Therefore the problem is closed. ϕ and p (pressure) are decoupled. ϕ can be solved independently first, and after it is obtained, the pressure p is evaluated.

$$p = f(\vec{v}) = f(\nabla\phi) \quad \rightarrow \quad \text{Solve for } \phi\text{, then find pressure.}$$

3.4 Bernoulli equation for potential flow (steady or unsteady)

Euler eq:

$$\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} = -\nabla\left(\frac{p}{\rho} + gy\right)$$

Substitute $\vec{v} = \nabla\phi$ into the Euler's equation above, which gives:

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\left|\nabla\phi\right|^2\right) = -\nabla\left(\frac{p}{\rho} + gy\right)$$

or

$$\nabla\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy\right\} = 0,$$

which implies that

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy = f(t)$$

everywhere in the fluid for unsteady, potential flow. The equation above can be written as

$$p = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right] + F(t)$$

which is the Bernoulli equation for unsteady or steady potential flow.

Summary: Bernoulli equation for ideal flow.

- Steady rotational or irrotational flow along streamline.

$$p = -\rho\left(\frac{1}{2}v^2 + gy\right) + C(\psi)$$

- Unsteady or steady irrotational flow everywhere in the fluid.

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right) + F(t)$$

- For hydrostatics, $\vec{v} \equiv 0$, $\frac{\partial}{\partial t} = 0$.

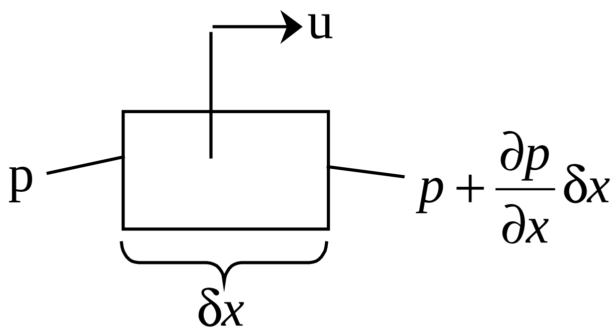
$$p = -\rho g y + c \leftarrow \text{ hydrostatic pressure (Archimedes' principle)}$$

- Steady and no gravity effect ($\frac{\partial}{\partial t} = 0, g \equiv 0$)

$$p = -\frac{\rho v^2}{2} + c = -\frac{\rho}{2}\left|\nabla\phi\right|^2 + c \leftarrow \text{Venturi pressure (created by velocity)}$$

- Inertial, acceleration effect

$$p \sim -\rho\frac{\partial\phi}{\partial t} + \cdots$$
$$\nabla p \sim -\rho\frac{\partial}{\partial t}\vec{v} + \cdots$$



3.5 - Boundary Conditions

- KBC on an impervious boundary

$$\underbrace{\vec{v}\cdot\hat{n}}_{\hat{n}\cdot\nabla\phi} = \underbrace{\vec{u}\cdot\hat{n}}_{U_n \text{ given}} \quad \text{no flux across boundary} \;\Rightarrow \frac{\partial \phi}{\partial n} = U_n \text{ given}$$

- DBC: specify pressure at the boundary, i.e.,

$$-\rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right) = \text{given}$$

Note: On a free-surface $p = p_{atm}$.

3.6 - Stream function

- continuity: $\nabla\cdot\vec{v} = 0$; irrotationality: $\nabla\times\vec{v} = \vec{\omega} = 0$

- velocity potential: $\vec{v} = \nabla\phi$, then $\nabla\times\vec{v} = \nabla\times(\nabla\phi) \equiv 0$for any ϕ, i.e. irrotationality is satisfied automatically. Required for continuity:

$$\nabla\cdot\vec{v} = \nabla^2\phi = 0$$

- Stream function $\vec{\psi}$ defined by

$$\vec{v} = \nabla\times\vec{\psi}$$

Then $\nabla\cdot\vec{v} = \nabla\cdot\left(\nabla\times\vec{\psi}\right) \equiv 0$ for any $\vec{\psi}$, i.e. satisfies continuity automatically.

Required for irrotationality:

$$\nabla\times\vec{v} = 0 \Rightarrow \nabla\times\left(\nabla\times\vec{\psi}\right) = \underbrace{\nabla\left(\nabla\cdot\vec{\psi}\right) - \nabla^2\vec{\psi}}_{\substack{\text{still 3 unknowns} \\ \vec{v} \longleftrightarrow \vec{\psi}}} = 0 \tag{1}$$

- For 2D and axisymmetric flows, $\vec{\psi}$ is a scalar ψ (so stream functions are more useful for 2D and axisymmetric flows).

  For 2D flow: $\vec{v} = (u, v, 0)$ and $\frac{\partial}{\partial z} \equiv 0$.

$$\vec{v} = \nabla \times \vec{\psi} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = \left(\frac{\partial}{\partial y}\psi_z\right)\hat{i} + \left(-\frac{\partial}{\partial x}\psi_z\right)\hat{j} + \left(\frac{\partial}{\partial x}\psi_y - \frac{\partial}{\partial y}\psi_x\right)\hat{k}$$

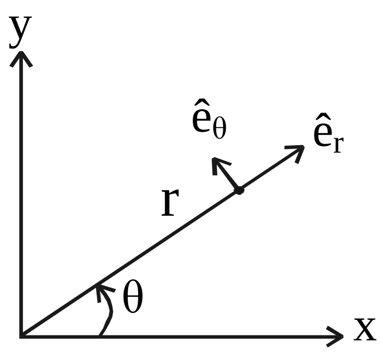
  Set $\psi_x = \psi_y \equiv 0$ and $\psi_z = \psi$, then $u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x}$

  So, for 2D:

$$\nabla \cdot \vec{\psi} = \frac{\partial}{\partial x}\psi_x + \frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z \equiv 0$$

  Then, from the irrotationality (see (1)) $\Rightarrow$ $\nabla^2 \psi = 0$ and ψ satisfies Laplace's equation.

- 2D polar coordinates: $\vec{v} = (v_r, v_\theta)$ and $\frac{\partial}{\partial z} \equiv 0$.



$$\vec{v} = \nabla \times \vec{\psi} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \psi_r & \psi_\theta & \psi_z \end{vmatrix} = \overbrace{\frac{1}{r}\frac{\partial \psi_z}{\partial \theta}}^{v_r}\hat{e}_r \overbrace{-\frac{\partial \psi_z}{\partial r}}^{v_\theta}\hat{e}_\theta + \overbrace{\frac{1}{r}\left(\frac{\partial}{\partial r}r\psi_\theta - \frac{\partial}{\partial \theta}\psi_r\right)}^{v_z}\hat{e}_z$$

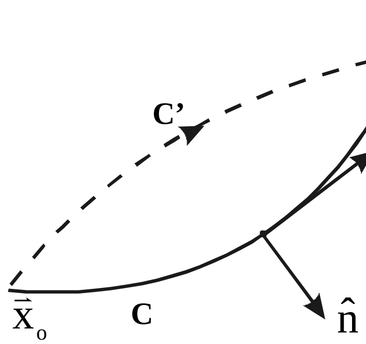
Again let $\psi_r = \psi_\theta \equiv 0$ and $\psi_z = \psi$, then $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ and $v_\theta = -\frac{\partial\psi}{\partial r}$.

- For 3D but axisymmetric flows, $\vec{\psi}$ also reduces to ψ (read JNN 4.6 for details).
- Physical Meaning of ψ.

  In 2D: $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$.

  We define

$$\psi(\vec{x},t) = \psi(\vec{x}_0,t) + \underbrace{\int_{\vec{x}_0}^{\vec{x}} \vec{v}\cdot\hat{n}dl}_{\substack{\text{total volume flux}\\ \text{from left to right}\\ \text{accross a curve C}\\ \text{between} \vec{x} \text{ and } \vec{x}_0}} = \psi(\vec{x}_0,t) + \int_{\vec{x}_0}^{\vec{x}} (udy - vdx)$$

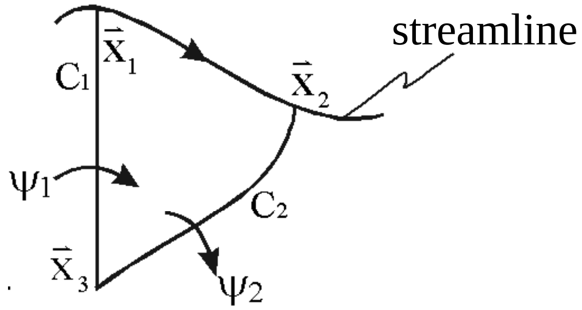


For ψ to be single-valued, $\int$ must be path independent.

$$\int_C = \int_{C'} \text{ or } \int_C - \int_{C'} = 0 \longrightarrow \oint_{C-C'} \vec{v}\cdot\hat{n}\; dl = \iint_S \underbrace{\nabla\cdot\vec{v}}_{continuity}\; ds = 0$$

Therefore, ψ is unique because of continuity.

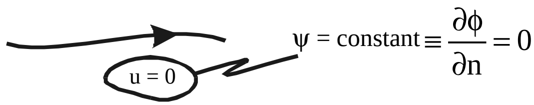
Let $\vec{x}_1, \vec{x}_2$ be two points on a given streamline ($\vec{v} \cdot \hat{n} = 0$ on streamline)

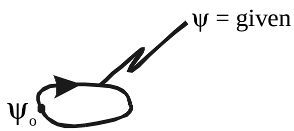


$$\underbrace{\psi\left(\vec{x}_2\right)}_{\psi_2} = \underbrace{\psi\left(\vec{x}_1\right)}_{\psi_1} + \int\limits_{\vec{x}_1}^{\vec{x}_2} \underbrace{\vec{v} \cdot \hat{n}}_{\substack{=0 \\ \text{along} \\ \text{streamline}}} dl$$

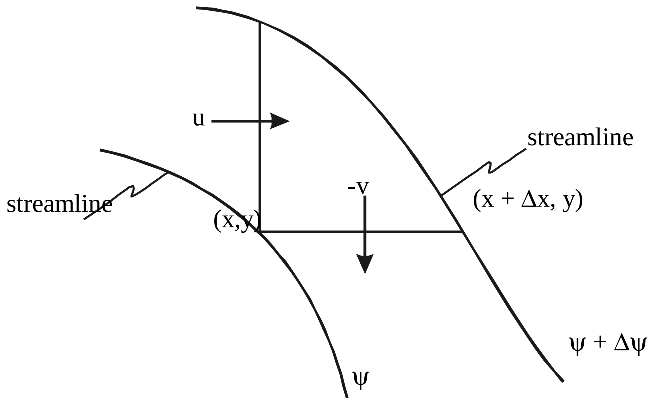
Therefore, $\psi_1 = \psi_2$,i.e., ψ is a constant along any streamline. For example, on an impervious stationary body $\vec{v} \cdot \hat{n} = 0$, so $\psi =$ constant on the body is the appropriate boundary condition. If the body is moving $\vec{v} \cdot \hat{n} = U_n$

$$\psi = \psi_0 + \int \underbrace{U_n}_{given} dl \quad \text{on the boddy}$$





Flux $\Delta\psi = -v\Delta x = u\Delta y$. Therefore, $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$



Summary: Potential formulation vs. Stream-function formulation for ideal flows

	potential	stream-function
definition	$\vec{v} = \nabla\phi$	$\vec{v} = \nabla \times \vec{\psi}$
continuity $\nabla \cdot \vec{v} = 0$	$\nabla^2\phi = 0$	automatically satisfied
irrotationality $\nabla \times \vec{v} = 0$	automatically satisfied	$\nabla \times \left(\nabla \times \vec{\psi}\right) = \nabla\left(\nabla \cdot \vec{\psi}\right) - \nabla^2\vec{\psi} = 0$
In 2D : $w = 0, \frac{\partial}{\partial z} = 0$		
	$\nabla^2\phi = 0$ for continuity	$\psi \equiv \psi_z : \nabla^2\vec{\psi} = 0$ for irrotationality
Cauchy-Riemann equations for (ϕ, ψ) = (real, imaginary) part of an analytic complex function of z = x +iy		
Cartesian (x, y)	$u = \frac{\partial\phi}{\partial x}$ $v = \frac{\partial\phi}{\partial y}$	$u = \frac{\partial\psi}{\partial y}$ $v = -\frac{\partial\psi}{\partial x}$
Polar (r,θ)	$v_r = \frac{\partial\phi}{\partial \mathrm{r}}$ $v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$	$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $v_\theta = -\frac{\partial\psi}{\partial r}$

For irrotational flow	use	ϕ
For incompressible flow	use	ψ
For both flows	use	ϕ or ψ

Given ϕ or ψ for 2D flow, use Cauchy-Riemann equations to find the other:

e.g. $\phi = \mathrm{xy}$ $\psi = ?$

$$\left.\begin{array}{l} \frac{\partial\phi}{\partial \mathrm{x}} = y = \frac{\partial\psi}{\partial y} \longrightarrow \psi = \frac{1}{2}y^2 + f_1(x) \\ \frac{\partial\phi}{\partial \mathrm{y}} = x = -\frac{\partial\psi}{\partial x} \longrightarrow \psi = -\frac{1}{2}x^2 + f_2(y) \end{array}\right\} \psi = \frac{1}{2}(y^2 - x^2) + const$$