13.42 READING 7:  
WAVE STATISTICS AND SPECTRA  

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1. 1/Nth HIGHEST MAXIMA

For design purposes it is useful to determine the occurrences of wave amplitude maxima above a certain level. Given a time-trace of wave height data from a buoy deployed in the ocean we can analyze it to determine such information. If we look at a given wave train, $y(t)$, over an interval in time, $T_o$ ($T_o$ is not necessarily a wave period but more like a duration of time which can consist of multiple periods), then within this interval of time there are a number of maxima, $a_1, a_2, a_3, a_4, \ldots, a_n$, where $a^{1/N}$ is the value exceeded by 1/N of the maxima. (Note: $a^{1/N}$ is not $a$ to the $1/N$ power)

EXAMPLE: Take $N = 3$; From a sequence of twelve measured wave heights find the $1/3$rd ($1/N$) highest wave height, $a^{1/3}$;

Measured wave heights: 6 5 3 4 7 11 8 9 5 4 2 5

There are 12 recorded wave maxima so there are four wave maxima above the $1/3$rd highest wave height. For this sequence $a^{1/3} = 6$ since it is the next highest wave maxima below 7. Had this been a infinitely long series of observed wave heights and 7 was the lowest value of the $1/3$rd highest wave maxima, then 6.99 might been a better estimation for this value. However in this short sequence the answer is 6.
Had the sequence been:

\[ \begin{bmatrix} 1 & 8 & 3 & 4 & 11 & 11 & 11 & 11 & 5 & 4 & 2 & 5 \end{bmatrix} \]

then \( a^{1/3} = 8 \), since the four highest wave heights are all equal to 11.

The probability of wave heights occurring above the 1/Nth highest wave is given by

\[
P(\eta \geq \eta_{1/N}) = \frac{1}{N} \approx \frac{2\sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}} e^{-(\eta_{1/N})^2/2}
\]

where

\[
\eta_{1/N} = \frac{a_{1/N}}{\sqrt{M_o}} = \sqrt{2 \ln \left( \frac{2\sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}} N \right)}
\]

The average value of ALL maxima above \( a^{1/N} \) is called the 1/N (Nth) highest average amplitude. This can be found using the formula for expected value of a variable:

\[
\overline{a_{1/N}} = E\left\{a_m\mid (a_m > a^{1/N})\right\}
\]

This is the expected value given \( a_m \) is greater than \( a^{1/N} \) as can be represented as

\[
\overline{a_{1/N}} = \int_{a^{1/N}}^{\infty} a \ p\left((a_m = a)\mid (a_m > a^{1/N})\right) \, da
\]

where the probability, \( p\left(a_m = a\mid a_m > a^{1/N}\right) \), is simply

\[
p\left(a_m = a\mid a_m > a^{1/N}\right) = \frac{P\left((a_m = a) \cap (a_m > a^{1/N})\right)}{P(a_m > a^{1/N})}
\]

Keeping the amplitude in non-dimensional form we can calculate the Nth highest average wave height using the approximate pdf given in the last reading.

\[
\eta_{1/N} \approx \frac{2N \sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}} \int_{\eta_{1/N}}^{\infty} \eta^2 e^{-\eta^2/2} \, d\eta_o
\]
For a value of $N = 3$, $a^{1/N}$ is considered the **significant wave amplitude** where $a^{1/N} \approx 2\sigma = 2\sqrt{\mathcal{M}_o}$ for $\varepsilon < 0.5$. The **significant wave height** is defined as twice the significant wave amplitude,

\begin{equation}
H^{1/3} = 2a^{1/3}.
\end{equation}

This value is very close to that which a casual observer would estimate as the wave height when watching the sea. This makes the significant wave height a very useful statistic. Maps of the significant wave height over the entire earth can be seen on satellite images. There are several links on the course webpage that illustrate this quantity over the earth surface. From such a satellite composite image we can see that the southern ocean is the most tumultuous ocean and has the highest significant wave height.

\section*{2. Ocean Wave Spectra}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{wave_energy_spectra.png}
\caption{Wave energy spectra. Red text indicates wave generation mechanisms and blue text indicates damping/restoring forces.}
\end{figure}
The majority of ocean waves are wind generated. Other wave generating mechanisms include earthquakes and planetary forces. Planetary forces drive tides and cause long period waves on the order of 12 to 24 hours. Earthquakes are the major cause of tsunamis which, while rare, can be catastrophic if the earthquake occurs near or on the coast.

Waves also encounter forces that tend to restore them to a flat surface. For small wavelength (high frequency) waves surface tension plays a large role in damping out these waves. The majority of waves are restored by gravity and longer period waves are damped by the coriolis force.

As wind begins to blow (between 0.5 - 2 knots) on a calm surface small ripples, capillary waves or “cat-paws”, tend to form. These small waves are on the order of less than 2 cm. As the wind becomes stronger wave amplitude increases and the waves become longer in order to satisfy the dispersion relationship. This growth is driven by the bernoulli effect, frictional drag, and separation drag on the wave crests.

Wind must blow over long periods of time and large distances to reach a fully developed sea state. When the phase speed of the wave crest matches the wind speed non-linear interactions stop (except friction) and the phase speed is maximized. The limiting frequency of the waves can be determined by the equation for phase speed and the dispersion relationship.

\[
C_p \approx U_w = \frac{\omega}{k} = \frac{g}{\omega}
\]

(2.1)

\[
\omega_c \approx \frac{g}{U_w}
\]

(2.2)

where \(U_w\) is the wind speed and \(\omega_c\) is the limiting frequency. Once wind stops viscosity erodes the waves slowly. The smallest wavelengths decay the fastest. Sample spectrum shapes are shown in figure 2.

For a storm with wind speed, \(U_w\), the effects of the storm can be felt at a distance from the storm, \(R\). The number of wave cycles between the storm and the observation location is \(N = R/\lambda\). The amplitude of the waves decays as \(e^{-\gamma t}\) where \(\gamma = 2\nu k^2 = 2\nu\omega^2/g^2\) (from Landau and Lifshitz).
The development of storms can be tabulated. Fetch is the length over which the wind must blow to have fully developed seas (given in standard miles), and the storm duration, given in hours, is the time the storm must last to result in a fully developed sea.

<table>
<thead>
<tr>
<th>Wind warnings</th>
<th>Beaufort scale</th>
<th>Wind speed (mph)</th>
<th>fetch (miles)</th>
<th>storm duration (hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>small craft</td>
<td>5-6</td>
<td>25</td>
<td>100</td>
<td>12</td>
</tr>
<tr>
<td>gale</td>
<td>7</td>
<td>35</td>
<td>400</td>
<td>28</td>
</tr>
<tr>
<td>hurricane</td>
<td>9</td>
<td>50</td>
<td>1050</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>3-4</td>
<td>12</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>

3. Typical Wave Spectra

Researchers have studying ocean waves have proposed several formulation for wave spectra dependent on a number of parameters (such as wind speed, fetch, or modal frequency). These formulations are very useful in the absence of measured data, but they can be subject to geographical and seasonal limitations.
Most ocean wave spectra take a standard form following the mathematical formulation:

\[ S^+(\omega) = \frac{A}{\omega^5} e^{-B/\omega^4} \]

The frequency peak is called the modal frequency. The area under the spectrum is the zeroth moment, \( M_0 \), which may be defined in terms of the significant wave height. For a narrow-banded spectrum the significant wave height is approximately four times the square root of the zeroth moment. Since the significant wave height depends on the wind speed, the spectrum could be formulated in terms of the wind speed instead of the significant wave height. While certain spectra can have more than one peak, it is assumed that a single storm produces a single-peaked spectrum and any second peak is due to a distant storm that sends waves to the considered location.

Several more specific theoretical representations of wave spectra have been developed using data collected by observation platforms and satellite data in various regions. These spectra are discussed below. When considering which spectrum formulation it is important to take into account the specific criteria that were used in developing the spectrum. Typically parameters that influence the spectrum are:

- Fetch limitations, i.e. whether the location we are considering has some physical boundaries that do not permit the waves to fully develop.
- Whether the seas are developing or decaying
- Seafloor topography: Deep water wave spectra are invalid in shallow waters, and vice versa. It may also be necessary to account for wave diffraction.
- Local currents: Strong currents may significantly impact the wave spectrum
- Presence of swells: Swells are waves that result from distant storms that travel a significant distance and arrive often at an angle that differs from the wind direction. If we use a spreading function to correct a unidirectional spectrum it will not account for the presence of swell. It is also important when measuring waves that the component that results from swell be accounted for separately.
The Pierson-Moskowitz spectrum (equation 3.2) was developed for fully developed seas in the Northern Atlantic Ocean generated by local winds.

\[
S^+ (\omega) = \frac{8.1}{10^3} \frac{g^2}{\omega^5} e^{-0.032 \left(\frac{g}{\zeta \omega^2}\right)^2}
\]

where \(\zeta\) is the significant wave height,

\[
\zeta \equiv H^{1/3} = 4 \sqrt{M_o},
\]

and \(\omega_m\) is the modal frequency,

\[
\omega_m = 0.4 \sqrt{g/\zeta}.
\]

This spectrum is developed under the following conditions: unidirectional seas, North Atlantic Ocean, fully developed local wind generation with unlimited fetch. The most critical of these assumptions is the fully developed assumption. For it is possible to achieve a larger heave response for a platform from a developing sea, even though the significant wave height may be smaller than that of a fully developed sea, since the modal frequency is higher and heave motions tend to have higher natural frequencies. In the case of a rolling ship the decaying sea might excite a larger roll motion since the natural frequency of roll tends to be relatively low.

In order to overcome the limitation of fully developed seas, a two parameter spectrum was developed. This spectrum is the Bretschneider spectrum (equation 3.5). The B-S spectrum replaced the Pierson-Moskowitz spectrum as the ITTC standard.

\[
S^+ (\omega) = \frac{1.25}{4} \frac{\omega_m^4}{\omega^5} \zeta e^{-1.25 \left(\frac{\omega_m}{\omega}\right)^4}
\]

where again, \(\zeta\) is the significant wave height,

\[
\zeta \equiv H^{1/3} = 4 \sqrt{M_o},
\]
If $\omega_m$ satisfies equation 3.4 then equation 3.5 reduces to equation 3.2. By allowing the user to specify the modal frequency and significant wave height, this spectrum can be used for sea states of varying severity from developing to decaying.

The Ochi Spectrum (equation 3.7) is a three parameter spectrum that allows the user to specify the significant wave height, the modal frequency, and the steepness of the spectrum peak.

\[
S^+(\omega) = \frac{1}{4} \left( \frac{4\lambda + 1}{\lambda} \right)^{\frac{\lambda}{\delta}} \frac{\zeta^2}{\omega^{4\lambda+1}} \exp \left\{ - \left( \frac{4\lambda + 1}{4} \right) \left( \frac{\omega_m}{\omega} \right)^{4\lambda+1} \right\}
\]

where $\Gamma(\lambda)$ is the gamma function, and $\lambda$ is the parameter that controls the spectrum steepness. For $\lambda = 1$, equation 3.7 reduces to equation 3.5. The Ochi spectrum is limited in that it also considers only unidirectional seas and unlimited fetch, but the designer can now specify the spectrum’s severity ($\zeta$), the state of development (peak frequency $\omega_m$) and isolate the important frequency range by dictating the spectrum width ($\lambda$). The ability to dictate $\lambda$ allows the designer to account for swell from a distant storm.

The JONSWAP spectrum (equation 3.8) was developed by the Joint North Sea Wave Project for the limited fetch North Sea and is used extensively by the offshore industry. This spectrum is significant because it was developed taking into consideration the growth of waves over a limited fetch and wave attenuation in shallow water. Over 2,000 spectra were measured and a least squares method was used to obtain the spectral formulation assuming conditions like near uniform winds.

\[
S^+(\omega) = \frac{a y^2}{\omega^{5}} e^{-\frac{\omega}{\omega_m}} \left( \frac{\omega}{\omega_m} \right)^{4\lambda \delta}
\]

where
\begin{align*}
\delta &= -\frac{(\omega - \omega_m)^2}{2\sigma^2 \omega_m^2} \\
\alpha &= 0.076 \pi^{-0.22} \\
\overline{\pi} &= \frac{gx}{U^2} \\
\sigma &= \begin{cases} 
0.07; \quad \omega \leq \omega_m \\
0.09; \quad \omega > \omega_m 
\end{cases}
\end{align*}

The wind speed in knots is $U$, $x$ is the fetch in nautical miles, and the modal frequency can be found as
\begin{equation}
\omega_m = 2\pi \ast 3.5 \ast (g/U) \pi^{-0.33}.
\end{equation}

To recap, in general for a narrow banded spectrum:
\begin{equation}
\int_{-\infty}^{\infty} S^\dagger(\omega) \, d\omega = M_o = \left(\frac{\zeta}{4}\right)^2
\end{equation}

We can account for the effects of two separate storms by adding the respective spectrums:
\begin{equation}
S^\dagger(\omega) = S_1^\dagger(\omega) + S_2^\dagger(\omega)
\end{equation}

We can also correct for unidirectionality multiplying the spectrum by a spreading function, $M(\mu)$,
\begin{equation}
S^\dagger(\omega, \mu) = S_{BS}^\dagger(\omega) \, M(\mu)
\end{equation}

where $M(\mu)$ spreads the energy over a certain angle contained within the interval $[-\pi, \pi]$ from the wind direction. The integral of $M(\mu)$ over this interval is one.
\[ \int_{-\pi}^{\pi} M(\mu) \, d\mu = 1 \]

For example we can choose a spreading function such that

\[ M(\mu) = \frac{2}{\pi} \cos^2 \mu \]

on the interval

\[ -\frac{\mu}{2} < \mu < \frac{\mu}{2} \]

4. Bretschneider Spectrum

To recap, the 15th International Towing Tank Conference (ITTC) in 1978 recommended using a form of the Bretschneider spectrum for average sea conditions when a more specific appropriate form of the wave spectrum is well defined. The general form of this spectrum is equation refeq:specgen.

\[ S^+(\omega) = \frac{A}{\omega^4} e^{-B/\omega^4} \]

The two parameters A and B are dependent on the modal frequency, \( \omega_m \), and the variance of the spectrum, \( M_o = (\text{rms})^2 = \sigma^2 \).

\[ \omega_m^4 = \frac{4}{5} B ; \quad B = 5 \omega_m^4 / 4 \]

\[ \text{Variance} = \sigma^2 = A/(4B) ; \quad A = 4\sigma^2 B \]

If we normalize the frequency, \( \omega \), by the modal frequency equation 4.1 becomes equation 4.4.

\[ S(\omega) = 5 \frac{\omega_m^4}{\omega^5} \sigma^2 e^{-\frac{5}{4} (\frac{\omega_m}{\omega})^4} \]
For a narrow banded spectrum, $\varepsilon < 0.6$, the significant wave height, $\zeta = H^{1/3} = 4\sqrt{M_o}$, where $M_o$ is the variance of the spectrum. For a wide banded spectrum, $\varepsilon = 1$, St. Denis (1980) showed that the significant wave height was approximately, $\zeta = 3\sqrt{M_o}$. This leaves us with the final form of the Bretschneider Spectrum.

$$S(\omega) = \frac{1.25}{4} \frac{\omega_m^4}{\omega^4} \zeta^2 e^{-1.25\left(\frac{\omega_m}{\omega}\right)^4}$$ \hspace{2cm} (4.5)

The moments of the spectrum can be calculated numerically. For simplification the following relationships have been given (see Principles of Naval Architecture vol. III for further discussion). The fourth moment diverges slowly as $\omega \to \infty$ thus the approximation helps analysis when calculation of this moment is necessary.

\begin{align*}
M_o &= \text{VARIANCE} = (RMS)^2 \\
M_2 &= 1.982 M_o \omega_m^2 = 1.982 \left(\frac{\zeta \omega_m}{4}\right)^2 \\
M_4 &\approx 7.049 M_o \omega_m^4 \quad \text{for frequencies less than } 5\omega_m
\end{align*}

5. Useful References

Read section four of the supplemental notes: Triantafyllou and Chryssostomidis, (1980) "Environment Description, Force Prediction and Statistics for Design Applications in Ocean Engineering"