

1)

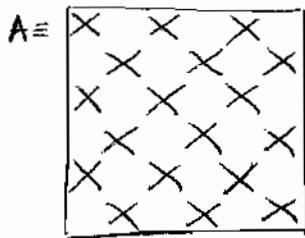
	1	2	3	4	5	6	← 2 nd DICE
1	2	3	4	5	6	7	
2	3	4	5	6	7	8	
3	4	5	6	7	8	9	
4	5	6	7	8	9	10	
5	6	7	8	9	10	11	
6	7	8	9	10	11	12	

↑
FIRST DICE

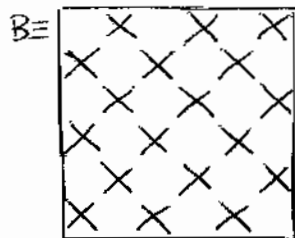
← VALUES OF THE SUM OF THE ROLL

PROBABILITY OF EACH COMBINATION EQUALS $\frac{1}{36}$.

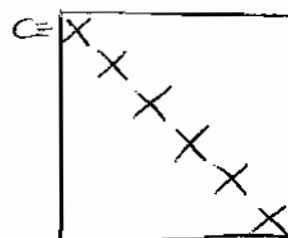
DEFINE EVENTS AS FOLLOWS:



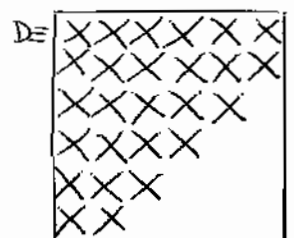
$P[A] = \frac{18}{36} = \frac{1}{2}$



$P[B] = \frac{18}{36} = \frac{1}{2}$

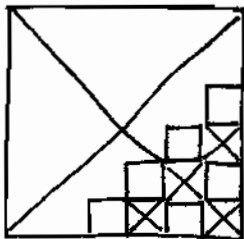


$P[C] = \frac{6}{36} = \frac{1}{6}$

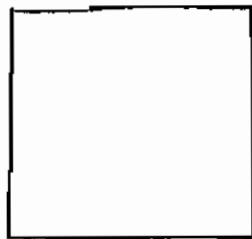


$P[D] = \frac{26}{36} = \frac{13}{18}$

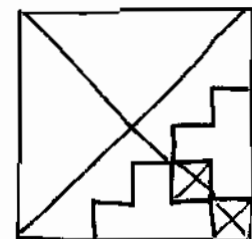
a) $P(A \cup D) = \frac{30}{36} = \frac{5}{6}$



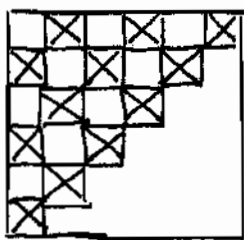
b) $P(B \cap C) = \emptyset$



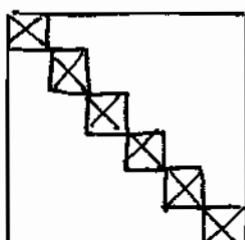
c) $P(C \cup D) = \frac{28}{36} = \frac{7}{9}$



d) $P(B \cap D) = \frac{12}{36} = \frac{1}{3}$



e) $P(A \cap C) = \frac{6}{36} = \frac{1}{6}$



2) P[FULL HOUSE]

WHERE FULL HOUSE = A FIVE CARD HAND IN WHICH THERE IS
A 3-OF-A-KIND AND A 2-OF-A-KIND

$$P[\text{FULL HOUSE}] = \frac{\# \text{ OF POSSIBLE FULL HOUSES}}{\text{TOTAL } \# \text{ OF POSSIBLE SCARD HANDS}}$$

FIRST LOOK AT THE TOTAL # OF POSSIBLE HANDS:

$$\text{CHOOSE 5 CARDS OUT OF 52 TOTAL} = \binom{52}{5} = \frac{52!}{47!5!}$$

$$\binom{52}{5} = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 52 \cdot 51 \cdot 49 \cdot 10 \cdot 2$$

NOW LOOK AT POSSIBLE FULL HOUSES:

THERE ARE 13 DIFFERENT FACE VALUES (e.g. A, 2, 3 etc.)
WHICH YOU CAN CHOOSE FOR THE 3 OF A KIND $\binom{13}{1}$

ONCE YOU'VE CHOSEN A FACE VALUE, THERE ARE FOUR
POSSIBLE SUITS ($\heartsuit, \diamondsuit, \clubsuit, \spadesuit$) SO THERE ARE $\binom{4}{3}$ POSSIBLE
3-OF-A-KINDS FOR EACH FACE VALUE

FOR EACH CHOICE OF FACE VALUE FOR THE 3-OF-A-KIND, THERE
ARE 12 POSSIBLE CHOICES FOR THE 2-OF-A-KIND $\binom{12}{2}$

ONCE AGAIN, THERE ARE FOUR SUITS TO CHOOSE THE
2 CARDS FROM SO THERE ARE $\binom{4}{2}$ POSSIBLE COMBINATIONS

THEREFORE THERE ARE $\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{2}$ POSSIBLE FULL HOUSES

$$\binom{13}{1} = \frac{13!}{12!1!} = 13 \quad \binom{4}{3} = \frac{4!}{3!1!} = 4 \quad \binom{12}{2} = \frac{12!}{11!1!} = 12 \quad \binom{4}{2} = \frac{4!}{2!2!} = 6$$

$$\therefore P[\text{FULL HOUSE}] = \frac{13 \cdot 4 \cdot 12 \cdot 6}{52 \cdot 51 \cdot 49 \cdot 10 \cdot 2} = \frac{6}{5 \cdot 17 \cdot 49} = \boxed{\frac{6}{4165}}$$

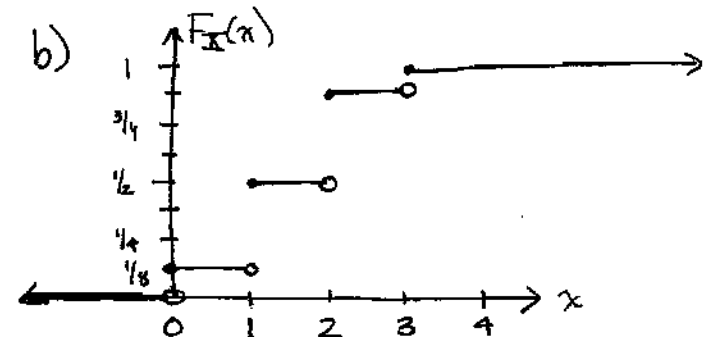
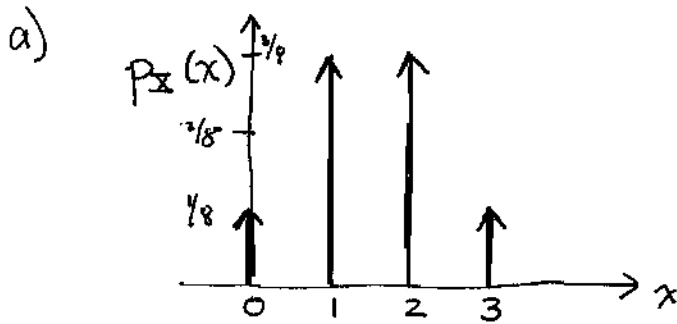
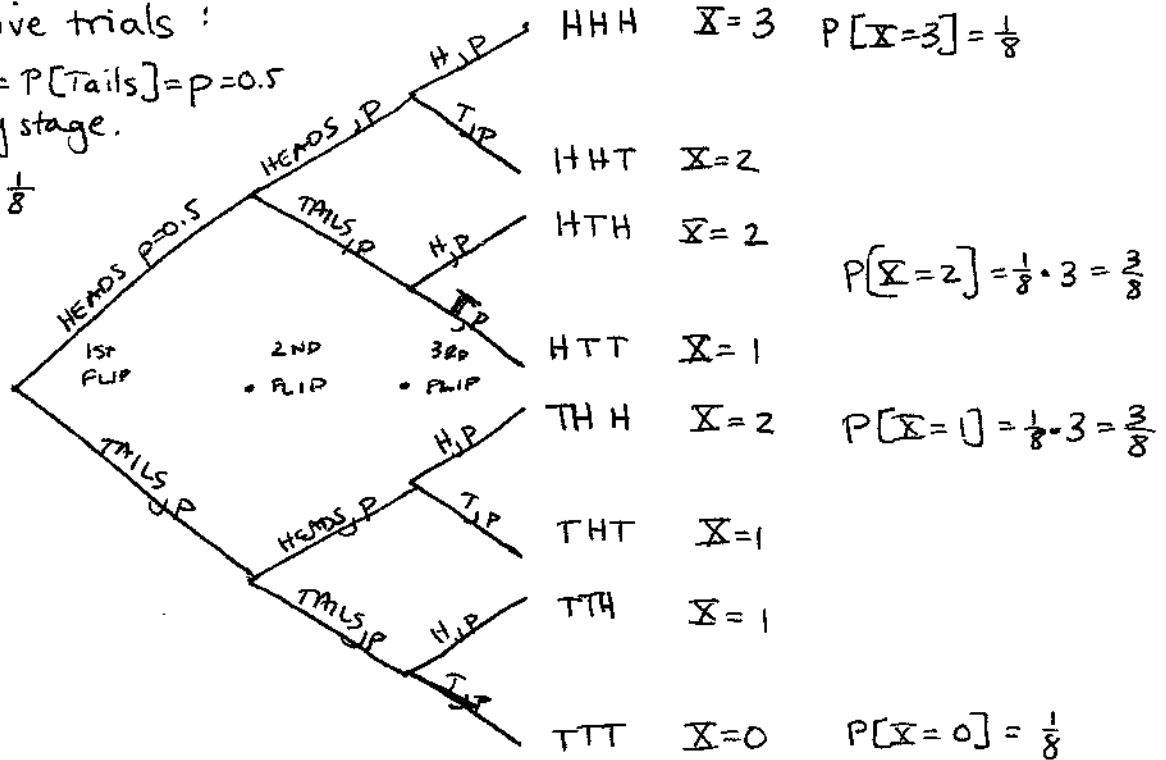
3) $X = \{\text{sum of 3 flips of a fair coin where heads}=1, \text{tails}=0\}$

3 successive trials :

$P[\text{Heads}] = P[\text{Tails}] = p = 0.5$
at every stage.

$(0.5)^3 = 0.125 = \frac{1}{8}$

(EACH OF THE 8 OUTCOMES HAS AN EQUAL PROBABILITY OF OCCURRING)



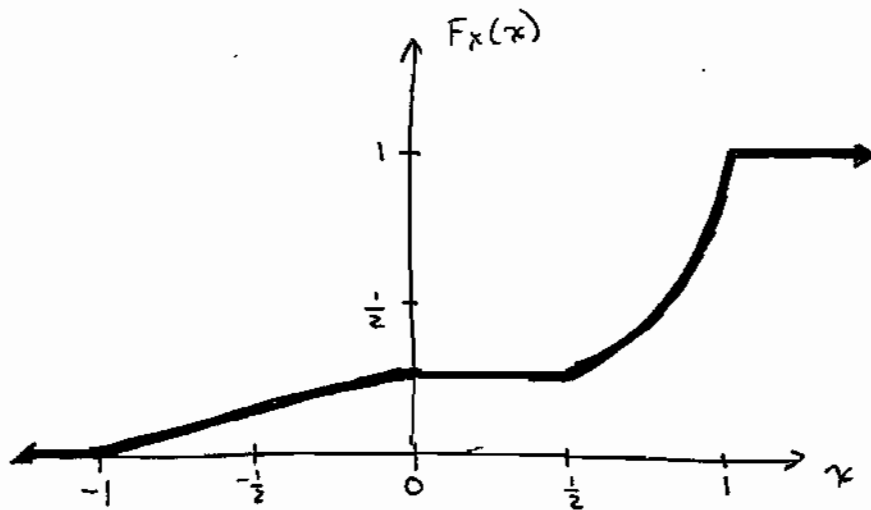
* Note there is a different scale on the $P_X(x)$ and $F_X(x)$ plots. *

c) $\mu_x = \sum_{n=0}^3 x \cdot p_x(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \boxed{1.5 = \mu_x}$

$\sigma_x^2 = \sum_{n=0}^3 (x^2 - \mu_x^2) p_x(x) = \sum_{n=0}^3 x^2 p_x(x) - \mu_x^2 = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} - (1.5)^2$
 $= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} - \frac{9}{4} = \frac{24}{8} - \frac{9}{4} = \frac{12}{4} - \frac{9}{4} = \frac{3}{4} \quad \boxed{0.75 = \sigma_x^2}$

$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = \boxed{0.866 = \sigma_x}$

4)



$$a) P(X \leq \frac{1}{2}) = F_X(\frac{1}{2}) = \boxed{\frac{1}{4}}$$

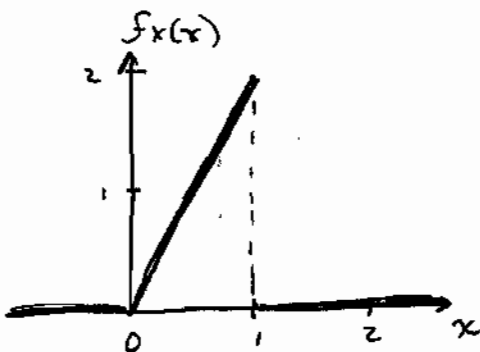
$$b) P(X \geq 0) = 1 - F_X(0) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$$

$$c) P(0 \leq X < \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(0) = \frac{1}{4} - \frac{1}{4} = \boxed{0}$$

$$d) P(X \leq \frac{3}{4}) = F_X(\frac{3}{4}) = (\frac{3}{4})^2 = \boxed{\frac{9}{16}}$$

$$e) P(X \geq 1) = 1 - F_X(1) = 1 - 1 = \boxed{0}$$

5)



$$a) \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$= \int_{-\infty}^0 x \cdot 0 dx + \int_0^1 x \cdot 2x \cdot dx + \int_1^{\infty} x \cdot 0 \cdot dx$$

$$= \int_0^1 2x^2 dx = 2 \frac{x^3}{3} \Big|_0^1 = \frac{2}{3} (1-0) = \boxed{\frac{2}{3} = \mu_X}$$

$$b) \sigma_X \Rightarrow \sigma_X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - (\mu_X)^2 = \int_0^1 x^2 \cdot 2x \cdot dx - (\frac{2}{3})^2 = \int_0^1 2x^3 dx - \frac{4}{9}$$

$$\left[\text{USE } \sigma_X^2 = E[X^2] - \mu_X^2 \right] = 2 \frac{x^4}{4} \Big|_0^1 - \frac{4}{9} = \frac{2}{4} (1-0) - \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18} = \sigma_X^2$$

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = \boxed{\frac{\sqrt{2}}{6} = \sigma_X}$$

$$6) \quad \eta(t) = A \cos(\omega t + \phi)$$

a) A is Gaussian with zero mean \therefore

$$f_A(a) = \frac{1}{\sigma_A \sqrt{2\pi}} e^{-\frac{a^2}{2\sigma_A^2}}$$

$$b) \quad p_d = \rho g A e^{kz} \cos(\omega t + \phi) \quad \text{at } z=0 \quad p_d = \rho g A \cos(\omega t + \phi)$$

c) To determine if $p_d(t)$ is a stationary process, you must check if the mean and variation and correlation are independent of time.

$$\mu_{p_d}(t) = E[p_d(t)] = E[\rho g A \cos(\omega t + \phi)]$$

$$= E[A] \cdot \rho g \cos(\omega t + \phi) = 0 \quad \therefore \mu_{p_d} = 0 = \text{constant}$$

↑
but I know that A has zero mean, i.e. $E[A] = 0$

$$\sigma_{p_d}^2(t) = E[(p_d(t) - \mu_{p_d})^2] = E[(p_d(t))^2] = E[(\rho g)^2 A^2 \cos^2(\omega t + \phi)]$$

$$= (\rho g)^2 \cos^2(\omega t + \phi) E[A^2] = (\rho g)^2 \cos^2(\omega t + \phi) \int_{-\infty}^{\infty} a^2 f_A(a) da$$

$$= (\rho g)^2 \cos^2(\omega t + \phi) \sigma_a^2 \quad \therefore \sigma_{p_d}^2 \text{ IS A FUNCTION OF TIME}$$

\therefore I know that it is NOT
A stationary process

$$7) \quad \eta(t) = \sum_{i=1}^N A_i \cos(\omega_i t + \phi_i)$$

FROM THE SPECTRUM PLOT AND THE EQ'N: $S_{\eta}(\omega) \delta\omega = \frac{1}{2} A_i^2$; $\delta\omega = 1 \frac{\text{rad}}{\text{sec}}$

$$\text{FOR } \omega_1 = 1 \frac{\text{rad}}{\text{sec}}, \quad \frac{1}{2} A_1^2 = (12)(1) \quad \text{so } A_1^2 = 24, \quad A_1 = 2\sqrt{6}$$

$$\text{FOR } \omega_2 = 2 \frac{\text{rad}}{\text{sec}}, \quad \frac{1}{2} A_2^2 = (18)(1) \quad \text{so } A_2^2 = 36, \quad A_2 = 6$$

$$\text{FOR } \omega_3 = 3 \frac{\text{rad}}{\text{sec}}, \quad \frac{1}{2} A_3^2 = (14)(1) \quad \text{so } A_3^2 = 28, \quad A_3 = 2\sqrt{7}$$

$$\text{FOR } \omega_4 = 4 \frac{\text{rad}}{\text{sec}}, \quad \frac{1}{2} A_4^2 = (8)(1) \quad \text{so } A_4^2 = 16, \quad A_4 = 4$$

Therefore:

$$\eta(t) = 2\sqrt{6} \cos(t + \phi_1) + 6 \cos(2t + \phi_2) + 2\sqrt{7} \cos(3t + \phi_3) + 4 \cos(4t + \phi_4)$$

USE RAND FUNCTION IN MATLAB TO GENERATE ϕ VALUES. IT IS OKAY TO EITHER USE THE RAND FUNCTION TO CREATE 40 VALUES OF ϕ : 4 ϕ 's IN EACH OF 10 EQUATIONS OR TO GENERATE 10 VALUES OF ϕ AND USE EACH ONE 4 TIMES IN EACH $\eta(t)$ EQUATION.

SEE ATTACHED MATLAB PLOTS.

- 8) a) The majority of sea waves are caused by wind.
- b) The phase speed of waves is maximized when it matches wind speed. Through the deep water dispersion relation, wavelength is limited for any frequency.
- c) Significant wave height is the average of the $\frac{1}{3}$ highest waves.
- d) Significant wave height correlates very closely to observations of wave height by experienced mariners. Therefore, historical observation data is usually significant wave height.

Homework 3 Problem 7: Sample MATLAB Code

```

t = 0:90;
w = [1 2 3 4];
A = [sqrt(24) sqrt(36) sqrt(28) sqrt(16)];
for j=1:10
for i=1:4
    phi = 2*pi*rand(1);
    %This randomly generates a separate phi value for each of the four
    %waves.
    WAVE(i,:) = A(i)*cos(w(i)*t+phi);
    %This wave equation iterates four times and generates waves for each of
    %the frequencies and their corresponding amplitudes.
end
S(j,:) = WAVE(1,:)+WAVE(2,:)+WAVE(3,:)+WAVE(4,:);
%This statement generates my wave elevation as a sum of the four waves
%generated above in each iteration. This loop iterates 10 times, created 10
separate
%realizations of the wave elevation equation with random phase shifts.
%
%
Mean_Ensemble = MEAN(S(:,30));
Variance_Ensemble = VAR(S(:,30));
Mean_Temporal(j,:) = MEAN(S(j,:));
Variance_Temporal(j,:) = VAR(S(j,:));
end
subplot(5,2,1), plot(t,S(1,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,2), plot(t,S(2,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,3), plot(t,S(3,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,4), plot(t,S(4,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,5), plot(t,S(5,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,6), plot(t,S(6,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,7), plot(t,S(7,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,8), plot(t,S(8,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,9), plot(t,S(9,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,10), plot(t,S(10,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
Mean_Ensemble
Variance_Ensemble
Mean_Temporal
Variance_Temporal

```

Example Output from Above Code (Note: It changes every run due to random realizations of phi)

HW37

Mean_Ensemble =

1.7494

Variance_Ensemble =

54.5025

Mean_Temporal =

0.0968
-0.0551
0.1645
0.0659
0.0887
-0.1158
-0.0079
-0.0449
-0.1013
-0.1384

Variance_Temporal =

52.7553
54.5923
51.9517
53.4987
52.7898
53.2102
51.6055
51.3866
52.4914
52.8724

diary off

Comments:

The ensemble statistics are computed over 10 data points, $S(:,30)$, which are each of the wave elevation realizations at time $t = 30$ seconds.

The temporal statistics are computed 10 times, for 10 separate wave elevation realizations, over 91 data points which represent a time range of $t = [0:90]$.

Therefore, the temporal statistics are more consistent from one execution of the m-file to the next.

Wave elevation, with a uniformly distributed ϕ over a 2π interval, is a stationary, ergodic random process. Therefore the ensemble statistics should equal the temporal statistics. Over multiple m-file executions, there will be more variability in the ensemble statistics, but they should approximate the temporal statistics over multiple iterations.

