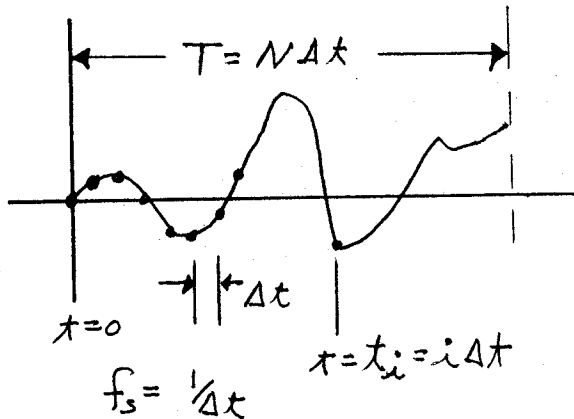


## The Discrete Fourier Transform

If a sampled function is assumed to be periodic, one may represent it as a Fourier series. Let  $x(t)$  be the sampled function, as depicted below.



where the total record length is  $T$ . Assume the continuous function  $x(t)$

is sampled at discrete points  $x(t_i)$  separated by a time interval  $\Delta t$ . The sampling frequency is  $f_s = \frac{1}{\Delta t}$ . The total number of samples  $N$  is related to the sampling frequency and record length:

$$f_s \cdot T = N = \frac{T}{\Delta t}$$

If  $x(t)$  is known for all  $t$  between 0 and  $T$ , then the Fourier series takes the form

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos 2\pi f_k t + b_k \sin 2\pi f_k t)$$

where

$$f_k = kf_1 = \frac{k}{T} \quad k = 1, 2, \dots, \infty$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi f_k t) dt \quad k = 0, 1, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi f_k t) dt \quad k = 1, 2, \dots$$

when working with digital data one does not know  $x(t)$  at all values of  $t$ , but only at discrete values of  $t_i$ , where  $x_i = x(t_i)$  and where  $t_i = i\Delta t$ ,  $i = 1$  to  $N$ ,  $T = N\Delta t$ . With knowledge at discrete points in time of  $x(t)$  one may find a discrete Fourier transform as follows.

## Summary of discrete Fourier transforms

$$\Delta f = \frac{1}{T} = f_1 \quad \text{resolution between frequency components}$$

$$f_{\max} = \frac{f_s}{2} \quad \text{Nyquist criterion}$$

$$f_s \cdot T = N \quad \text{the number of sample points}$$

$$T = \text{record length}$$

Because a discrete Fourier transform gives you information only up to  $\frac{f_s}{2}$  your measured signal must not contain components greater than  $\frac{f_s}{2}$ . You must filter out higher frequency components or sample at a greater  $f_s$ . The coefficients of the discrete Fourier series terms are given by

$$a_0 = \frac{2}{N} \sum_{i=1}^N x_i$$

$$a_n = \frac{2}{N} \sum_{i=1}^N x_i \cos\left(n \frac{2\pi}{T} t_i\right)$$

$$b_n = \frac{2}{N} \sum_{i=1}^N x_i \sin\left(n \frac{2\pi}{T} t_i\right)$$

The time series may be reconstructed at discrete points in time as a sum of sine and cosine terms given by

$$x_i = \frac{a_0}{2} + \sum_{n=1}^{N/2} a_n \cos\left(n \frac{2\pi}{T} t_i\right) + b_n \sin\left(n \frac{2\pi}{T} t_i\right)$$

Note the maximum frequency component in this Fourier series expression for  $x(t)$  is at

$$n = \frac{N}{2} \Rightarrow$$

$$\omega_{\max} = \frac{N}{2} \frac{2\pi}{T} = \frac{N}{2} \Delta\omega = \frac{N}{2} \omega_1 = \pi f_s$$

$$f_{\max} = \frac{N}{2} \cdot \frac{1}{T} = \frac{N}{2} \Delta f = \frac{N}{2} f_1 = \frac{f_s}{2}$$

If  $v(t)$  is the displacement of a mass-spring, single degree of freedom oscillator, then

$$\text{Kinetic Energy} = \frac{1}{2} m v^2$$

$$\text{Potential Energy} = \frac{1}{2} k v^2$$

$$\text{Average Energy} = \frac{1}{T} \int_0^T \left( \frac{1}{2} m \dot{v}^2 + \frac{1}{2} k v^2 \right) dt \quad \text{as } T \rightarrow \infty$$

$$\text{if } v(t) = \sum_n a_n \cos \omega_n t + b_n \sin \omega_n t$$

Then every term inside the integral will be products of sines and cosines with one of the following forms

$$a_i a_j \cos \omega_i t \cdot \cos \omega_j t$$

$$a_i b_j \cos \omega_i t \sin \omega_j t$$

$$b_i b_j \sin \omega_i t \sin \omega_j t$$

The only non-zero results will be from  $i = j$ , which yields terms of the form

$$\frac{a_i^2}{2} \cos^2(\omega_i t) + \frac{b_i^2}{2} \sin^2(\omega_i t)$$

At any single frequency,  $n = i$ , the total vibration energy will be proportional to  $\frac{a_i^2 + b_i^2}{2}$ . If we plot these values for all frequency components, we have the spectrum.

### Spectrum of a signal $x(t)$

The discrete power spectral density of  $x(t)$  is a plot of the mean square value at each frequency divided by the  $\Delta f$  between points

$$S_{xx}(f) = \frac{a_n^2 + b_n^2}{2\Delta f} \quad (\text{units of } \frac{x^2}{\text{Hz}})$$

where, as shown above,  $a_n$  and  $b_n$  are the coefficients of the discrete Fourier transform of  $x(t)$

For a discrete transform it is invalid to use results at frequencies greater than  $\frac{f_s}{2}$  which is known as the Nyquist criterion. This is because information in the signal at frequencies greater than  $\frac{f_s}{2}$  will be incorrectly interpreted at the wrong frequency, a problem known as aliasing.

Before sampling the data one must filter it to remove frequency components greater than  $\frac{f_s}{2}$ .

As a practical matter the sampling frequency needs to be five or six times greater than the highest frequency of interest in the signal, so that the filter does not alter the data.