**PROBLEM 1**

Compare the torsional and axial wave propagation speeds of 60mm diameter bars of

1. **Steel** \( (E = 2.1 \times 10^{12} \text{ dynes/cm}^2, \rho = 7.8 \text{ g/cm}^3) \)
2. **Aluminum** \( (E = 0.7 \times 10^{12} \text{ dynes/cm}^2, \rho = 2.7 \text{ g/cm}^3) \)
3. **Magnesium** \( (E = 0.41 \times 10^{12} \text{ dynes/cm}^2, \rho = 1.8 \text{ g/cm}^3) \)

To calculate the torsional wave propagation velocities we need the shear modulus \( G \) for each material. This is related to Young’s modulus by

\[
G = \frac{E}{2(1 + \nu)}
\]

where \( \nu \) is the Poisson’s ratio, equal to 0.32 for steel, 0.33 for aluminum and 0.35 for magnesium. The results are obtained straightforward, and are shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Axial velocity ( c_{\text{axial}} = \sqrt{\frac{E}{\rho}} ) (cm/sec)</th>
<th>Torsional velocity ( c_{\text{axial}} = \sqrt{\frac{G}{\rho}} ) (cm/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>518,875</td>
<td>319,345</td>
</tr>
<tr>
<td>Aluminium</td>
<td>509,175</td>
<td>312,195</td>
</tr>
<tr>
<td>Magnesium</td>
<td>477,261</td>
<td>290,452</td>
</tr>
</tbody>
</table>

You can see that, while Young’s modulus varies between metals, the mass density varies proportionally. Generally, the axial wave propagation velocity in metals is around 5,000m/sec while the torsional is around 3,000m/sec.
PROBLEM 2
Find the natural frequencies and mode shapes for longitudinal vibration of a magnesium bar of length \( l = 0.8 \text{m} \) and boundary conditions:

1. Clamped – Clamped
2. Clamped – Free

By the same way that we have done so far (i.e. by examining dynamic equilibrium of a differential element of the rod), we obtain the wave equation which reads:

\[
c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}
\]

where

\[
c = \sqrt{\frac{E}{\rho}}
\]

Again, we seek a solution of the form \( u(x,t) = U(x) \cdot T(t) \), i.e. we separate the variables. The solution will read

\[
U(x) = A \cos \frac{o_x}{c} + B \sin \frac{o_x}{c}
\]

\[
T(t) = C \cos o_t t + D \sin o_t t
\]

Function \( U \) will give us the mode shapes of vibration. Since \( U \) depends only on \( x \), the constants \( A \) & \( B \) will only depend on the boundary conditions. On the other hand, \( T \) is only a function of time and therefore constants \( C \) & \( D \) will be associated with the initial conditions. In this exercise we will only deal with \( U \), the boundary conditions and obtain the mode shapes for each case.

**Clamped – Clamped**
The boundary conditions imply:

\[
U(0) = U(l) = 0 \Rightarrow \begin{cases} A = 0 \\ B \sin \frac{o_l}{c} = 0 \Rightarrow \sin \frac{o_l}{c} = 0 \end{cases}
\]

Since the boundary conditions must always be satisfied, it must be \( \sin \frac{o_l}{c} = 0 \), which implies \( o_l = \frac{n \pi c}{l}, n = 1, 2, \ldots \). Plugging this into \( U \) yields the mode shape equation, \( U(x) = B \sin \frac{n \pi x}{l} \). Plugging in the numbers gives us \( \omega_n = 18.739n \), and \( U(x) = B \sin(3.925nx) \).
Clamped – Free
The boundary conditions imply:

\[ U(0) = 0 \Rightarrow A = 0 \]
\[ \sigma(l) = 0 \Rightarrow E \varepsilon(l) = 0 \Rightarrow \varepsilon(l) = 0 \Rightarrow \frac{dU}{dx} \bigg|_{x=l} = 0 \Rightarrow B \frac{\omega}{c} \cos \frac{\omega x}{c} = 0 \Rightarrow \cos \frac{\omega x}{c} = 0 \Rightarrow \omega_n = \frac{(2n+1)\pi c}{2l}, n = 0, 1, 2, \ldots \]

Plugging these natural frequencies in \( U \) we obtain

\[ U(x) = B \sin \left( \frac{(2n+1)\pi x}{2l} \right) = B \sin(1.9625(2n+1)x). \]

As mentioned, the mode shapes are inherent of the structure and its boundary conditions (e.g. rod clamped at both ends). One can directly use the results, i.e. the natural frequencies and mode shapes, without having to derive them every time one wants to solve a vibration problem.

PROBLEM 3
On infinitely long bars with the properties given in problem 1, find the wave length of 250Hz axial and torsional waves.

The wavelength is given by \( \lambda = \frac{c}{f} \). Therefore,

<table>
<thead>
<tr>
<th></th>
<th>Axial wavelength</th>
<th>Torsional wavelength</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \lambda = \frac{c}{f} ) (cm)</td>
<td>( \lambda = \frac{c}{f} ) (cm)</td>
</tr>
<tr>
<td>Steel</td>
<td>2075</td>
<td>1277</td>
</tr>
<tr>
<td>Aluminium</td>
<td>2036</td>
<td>1248</td>
</tr>
<tr>
<td>Magnesium</td>
<td>1909</td>
<td>1161</td>
</tr>
</tbody>
</table>

PROBLEM 4
Find the phase velocity of transverse waves travelling on a synthetic rope, 10mm in diameter, with a density of 1.0g/cm\(^3\), and under a tension of 4000N. What is the wave length of 100Hz waves?

The solution is again straightforward:

\[ c = \sqrt{\frac{T}{\rho I}} = \sqrt{\frac{T}{\frac{1.0 g}{cm^3} \pi (0.5 cm)^2}} = \sqrt{\frac{4000N}{0.0785 kg/m}} = 225.7 m/s \]
\[ \lambda = \frac{c}{f} = \frac{225.7}{100} = 2.25 m \]
PROBLEM 5
For the cable in problem 4 above, what are the natural frequencies and mode shapes of a piece 50m in length with fixed ends?

Following the same procedure as in problem 2, we find that

\[ \omega_n = \frac{n\pi c}{l} = 14.18n, \quad n = 1, 2, \ldots \]

\[ U_n(x) = \sin \frac{n\pi x}{l} = \sin(0.063nx) \]

PROBLEM 6
Let the cable described above be excited by a distributed force \( f(x) \).

\[ f(x) = \begin{cases} f_0 \cos(\omega t); & 0 < x \leq \frac{l}{2} \\ -f_0 \cos(\omega t); & \frac{l}{2} < x \leq l \end{cases} \]

Find the modal force for mode 2 and for all the odd modes, i.e. 1, 3, 5,...

Eye-balling the problem: Observe that the exciting function is anti-symmetric, just like all the even modes. We should expect then, that only the anti-symmetric modes are excited, and all the symmetric mode responses are zero. Furthermore, the spatial distribution of the exciting force “looks like” the second mode shape. So, we should expect that a major contribution to the response should come from the second mode.

In deed, the modal forces will read

\[ Q_n(t) = \int_0^l f(x)U_n(x)dx = \int_0^{l/2} f_0 \cos(\omega t) \sin \frac{n\pi x}{l} dx + \int_{l/2}^l (-f_0) \cos(\omega t) \sin \frac{n\pi x}{l} dx \]

\[ = \left( -2\cos \frac{n\pi}{2} + \cos n\pi + 1 \right) \frac{f_0 l}{n\pi} \cos \omega t \]

For \( n \) odd, \( Q_n(t) = 0 \). For \( n = 2 \), \( Q_2(t) = \frac{2f_0 l}{\pi} \cos \omega t \), which is the modal forcing function for mode 2. The result would have been the same (and in fact the algebra would have been exactly the same too), if we had chosen to analyse the forcing function in Fourier series.
PROBLEM 7
What is the steady state response of the cable from mode 2 only, if the excitation frequency is
1. \( \omega = \omega_2 \)
2. \( \omega = \omega_1 \)

Let the damping of the second mode be 2%.

We have already obtained the modal force for mode 2. The modal mass will be

\[
\mu_2 = \int_0^l \rho_l (x) U_2^2 (x) \, dx = \rho_l \int_0^l U_2^2 (x) \, dx = \rho_l \frac{l}{2} = 1.962 \text{ kg}
\]

and then, the modal stiffness will be \( \kappa_2 = \mu_2 \omega_2^2 = 1578 \text{ N/m} \). If the excitation frequency is the same as the 2nd natural frequency, mode 2 will be driven to resonance. The response will be

\[
q_2 = Q_2 \left| H_{q_2/q_2} \right| \cos(\omega_2 t - \varphi) = \frac{2 f_0 l}{\pi} \frac{1}{\kappa_2} \frac{1}{2 \zeta_2} \cos(\omega_2 t - \frac{\pi}{2}) = \frac{2 f_0 l}{\pi} 0.0158 \cos(\omega_2 t - \frac{\pi}{2})
\]

If the excitation frequency is equal to the 1st natural frequency, the response will be

\[
q_2 = Q_2 \left| H_{q_2/q_2} \right| \cos(\omega_1 t - \varphi) = \frac{2 f_0 l}{\pi} \frac{1}{\kappa_2} \frac{1}{\sqrt{1- \left( \frac{\omega_1}{\omega_2} \right)^2}} \cos(\omega_1 t - \varphi) = \frac{2 f_0 l}{\pi} 0.00254 \cdot 1.333 \cos(\omega_1 t - \varphi)
\]

where \( \varphi = \tan^{-1} \left( \frac{2 \zeta_2 \omega_1}{\omega_2} \right) \)

\[\begin{align*}
2 \zeta_2 \omega_1 \\
\omega_2
\end{align*}\]