PROBLEM 1
Find the lowest natural frequencies for cantilever beams of length 3 ft and the following cross sections. Require that the material cross sectional area of each be 1 in\(^2\). Assume the material is steel with a specific gravity 7.5 and Young’s modulus of \(E = 30\times10^6\) psi. What is the speed of sound in the material? What is the speed of bending waves for each case?

Knowing that \(A = 1.0\) in\(^2\), we calculate the dimensions of the beam in each case. From there, we calculate the moment of inertia for each case:

- **Square cross section**: \(I = \frac{b^4}{12}\)
- **Rectangular cross section**: \(I = \frac{bh^3}{12}\)
- **Circular cross section**: \(I = \frac{\pi r^4}{4}\)
- **Ring cross section**: \(I = \frac{\pi r_{out}^4}{4} - \frac{\pi r_{in}^4}{4}\)

Following that, \(\kappa = \sqrt{\frac{I}{A}}\) is calculated for each cross section. The lowest natural frequency is then evaluated as \(\omega_n = \frac{\pi^2 C_T \kappa}{4L^2}(1.194)^2\), and the velocity of the transverse waves will be \(C_T = \sqrt{\omega_n \kappa C_L}\). The results for each case are show in the table below.
### Dimensions

<table>
<thead>
<tr>
<th>Shape</th>
<th>Square</th>
<th>Rectangular</th>
<th>Circular</th>
<th>Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions</td>
<td>( b = 1 )</td>
<td>( h = \frac{1}{\sqrt{3}}, c = \sqrt{3} )</td>
<td>( r = \frac{1}{\sqrt{\pi}} )</td>
<td>( d_{\text{out}} = \frac{4}{\sqrt{3}\pi} )</td>
</tr>
<tr>
<td></td>
<td>( d_{\text{in}} = \frac{2}{\sqrt{3}\pi} )</td>
<td>( \frac{5}{12\pi} )</td>
<td></td>
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</table>

| Moment of Inertia | \( I = \frac{1}{12} \) | 0.0278 | 0.0795 | 5/12\pi |
| \( \kappa = \sqrt{\frac{I}{A}} = \frac{1}{\sqrt{2}\sqrt{3}} \) | 0.1667 | 0.2821 | 0.3641 |
| \( \omega_n = \frac{\pi^2 C_L \kappa (1.194)^2}{4L^2} \) | 159.84 | 92.2831 | 156.2 | 201.65 |
| \( C_T = \sqrt{\omega_n \kappa L} \) | 3068.04 | 1771.33 | 2998.11 | 3870.54 |

### PROBLEM 2

A steel wire is composed of two equal sections, whose diameters differ by a factor of two. If the diameter of the smaller section is 0.1in, the total length is 50ft, and the tension is 300lbs, estimate, using the Rayleigh quotient, the lowest natural frequency of the wire.

For this problem I used Mathematica. So, not being reluctant to complicate the calculus, I chose a trial function other than sine, that I assumed would be closer to the actual mode shape. First I “estimated” the mode shape values at four points:

\[
\begin{align*}
  w & = 0 @ x = 0 \\
  w & = 0.6 @ x = 0.3L \\
  w & = 1.0 @ x = 0.6L \\
  w & = 0 @ x = L
\end{align*}
\]

Using a polynomial interpolation (Lagrange), I derived a polynomial that has the values above at the corresponding points. Since I used the real \( x \) dimension (instead of a normalized one), I expect the polynomial coefficients to be very small. Indeed, the trial function is

\[
F(x) = -2.02087 \times 10^{-8} x^3 + 7.82628 \times 10^{-6} x^2 + 0.00257937 x
\]

Below is a plot of \( F \) versus \( x \) for \( x = 0 \ldots 600 \) in.
The cross-sectional area for each of the two segments will read,

\[ A_1 = \frac{\pi d_1^2}{4} = 0.007854 \text{in}^2 \]
\[ A_2 = \frac{\pi d_2^2}{4} = 0.003142 \text{in}^2 \]

and the mass/length will then be

\[ \rho_{1,1} = A_1 \rho = 0.002127 \text{lbs/in} \]
\[ \rho_{1,2} = A_2 \rho = 0.00851 \text{lbs/in} \]

where

\[ \rho = \frac{468 \text{lbs/ft}^2}{\text{in}^3} = 0.27 \text{lbs/in}^3 \]

The energy formulation of the Rayleigh quotient will then read

\[ \lambda_r = \frac{\int_0^{L/2} T \left( \partial_x F(x) \right)^2 \, dx}{\int_0^{L/2} \rho_{1,1} F^2(x) \, dx + \int_{L/2}^L \rho_{1,2} F^2(x) \, dx} = ... = 1.49 \]

If you use a sine trial function the rest of the formulation is identical to this one. The estimation of the natural frequency would be \( \lambda_r = 1.55 \).
PROBLEM 3

A bridge 100ft long is constructed of steel. The cross sections look approximately as shown in the figure below.

a. Obtain an estimate of the lowest natural frequency of the bridge. State your assumptions

b. Over the years asphalt is added to the roadway such that the mass per unit length increases by 30%. How is the natural frequency affected?

Note the following assumptions:

- The bridge is simply supported at the ends. This is often true for bridges – they are statically determined!
- The cross beam does not provide any stiffness to the vertical oscillation. All stiffness in the vertical sense is provided by the side – girders. This is also reasonable, first because of the small height of the cross beam, second because this is usually not continuous along the length in steel bridges. (See the figure below, where only one side-girder and only the half of the cross beams are shown).

Furthermore, the bridge can oscillate vertically and horizontally, with corresponding mode shapes in each direction. The lowest natural frequency will be for the first mode of the vertical oscillation, since it is in the vertical direction that the cross section has the smallest stiffness (try picturing the cross section as follows: Horizontally, as a high I-beam, and vertically as a low-depth H beam).

Let \( I \) denote the moment of inertia of one girder. Then, the overall stiffness of the bridge in the vertical direction will be \( 2EI \), \( E \) being the Young’s modulus for steel. The first natural frequency will be

\[
\omega_n = \frac{\pi}{L} \sqrt{\frac{2EI}{\rho A}}
\]

If the mass increases by 30%, the new natural frequency will be

\[
\omega'_n = \frac{\pi}{L} \sqrt{\frac{2EI}{1.3\rho A}} = 0.877 \frac{\pi}{L} \sqrt{\frac{2EI}{\rho A}} = 0.877\omega_n
\]
**PROBLEM 4**

Use the Rayleigh Quotient to obtain an estimate of the first natural frequency of the uniform cantilever beam shown in the figure below. Assume $A$, $E$, $\rho$ & $I$ are constant. Use $f(x) = ax^2$ as the trial function. State which form of the trial function you use, and show that it satisfies all the necessary boundary conditions. In what way is your estimate bounded?

I will use the “energy” formulation of the Rayleigh quotient for the given trial function. First, I check that the trial function satisfies all the kinematic boundary conditions. Indeed,

Zero displacement @ $x = 0$: $f(x = 0) = ax^2|_{x=0} = 0$
Zero rotation @ $x = 0$: $\partial_x f(x = 0) = 2ax|_{x=0} = 0$

Also, the trial function must have non-zero second derivative, which is satisfied as $\partial_{xx}f(x) = 2a \neq 0$.

Application of Rayleigh’s quotient to this trial function yields

$$\lambda_r = \frac{\int_0^L EI(\partial_{xx} f(x))^2 \, dx}{\int_0^L \rho Af^2(\chi) \, d\chi} = \frac{20EI}{\rho AL^4}$$

The true natural frequency is known to be $\omega_n^2 = \pi^4 \frac{EI}{16\rho AL^4}$. Our error in estimating the natural frequency is consequently $\frac{\sqrt{\lambda_r} - \omega_n}{\omega_n} = \frac{\sqrt{20} - \frac{\pi^2}{4}}{\frac{\pi^2}{4}} = 81\%$

**PROBLEM 5**

Find the equation of motion for the bending vibration of a beam on a uniform elastic foundation. The foundation has a spring constant $K_f$ which has units of force per unit length per unit deflection.

Note: This “elastic foundation”, when used for soils, is known as the Winkler model for elastic soils and is used extensively in geomechanical practice. The following figure may clarify the model:
Dynamic equilibrium of vertical forces on the above differential element of the beam requires

\[-(V + dV) - F + V - K_f \, w \, dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \Rightarrow\]
\[-dV - Fdx - K_f \, w \, dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \]

Noting that \( dV = \frac{\partial V}{\partial x} dx; dM = \frac{\partial M}{\partial x} dx \) and \( V = \frac{\partial V}{\partial x} \), the relationship above becomes

\[-\frac{\partial V}{\partial x} dx - Fdx - K_f \, w \, dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \Rightarrow\]
\[-\frac{\partial^2 M}{\partial x^2} - F - K_f \, w = \rho A \frac{\partial^2 w}{\partial t^2} \Rightarrow\]
\[\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + F + K_f \, w \, dx = -\rho A dx \frac{\partial^2 w}{\partial t^2} \]

Assuming free vibration, \( F = 0 \), and constant section and material properties, the equation above becomes

\[EI \frac{\partial^4 w}{\partial x^4} + K_f \, w = -\rho A \frac{\partial^2 w}{\partial t^2} \]

The equation above is the equation of motion. It can be solved by separating the variables such that \( w(x,t) = W(x)T(t) = W(x)e^{i\omega t} \).

**PROBLEM 6**

For the case that the beam in problem 5 is pinned at the ends, and has a total length of \( L \), show that the mode shapes are given by \( B_n = \sin \frac{n\pi x}{L} \), and find an expression for the natural frequencies.

Separating the variables as shown and plugging into the equation of motion yields

\[EI \frac{\partial^4 W(x)}{\partial x^4} + K_f W(x) = \rho A \omega_n^2 W(x)\]

Substituting \( W(x) = B_n = \sin \frac{n\pi x}{L} \) yields

\[EI \left( \frac{n\pi}{L} \right)^4 \sin \frac{n\pi x}{L} + K_f \sin \frac{n\pi x}{L} = \rho A \omega_n^2 \sin \frac{n\pi x}{L} \Rightarrow\]
\[EI \left( \frac{n\pi}{L} \right)^4 + K_f = \rho A \omega_n^2 \]

which is satisfied for \( \omega_n^2 = \frac{EI}{\rho A} \left( \frac{n\pi}{L} \right)^4 + \frac{K_f}{\rho A} \). These are that natural frequencies of the beam.