Review Question 1.15
Periodic motion is any motion that repeats itself in equal time intervals. Harmonic motion can be described in one dimension by an expression of \( x(t) = x_0 \sin(\omega t - \varphi) \).

Problem 1.9
Assuming that the upper and lower parts of the shaft are fixed, and the degree of freedom of the system is the rotation of the wider ring (of radius \( R \)), the solution is obtained by first identifying the torsional springs that are connected in parallel or in series. The linear counterpart of the assembly would look like:

![Diagram of torsional springs](image)

The corresponding \( K \)’s will be

\[
K_{1,2,3} = \left[ \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \right]^{-1}
\]

\[K_4 = k_4\]

\[K_5 = k_5 \cdot r \times r = k_5 \cdot r^2\]

\[K_6 = k_6 \cdot r \times r = k_6 \cdot r^2\]

And the overall equivalent stiffness will be

\[K_{eq} = K_{1,2,3} + K_4 + K_5 + K_6\]
Problem 1.13
We choose the degree of freedom by which we describe the motion of our system to be the rotation $\theta$ about the rotational spring. The equivalent spring constant for the system is found by imposing a unit rotation $\theta = 1$ and determining the moment required for that rotation.

Combining the springs in the same way as in the previous problem we obtain

$$K_{eq} = \left(\frac{1}{k_2} + \frac{1}{k_3}\right)^{-1} l_1^2 + k_1 l_1^2 + k_2 l_2^2 + k_{buoyancy} l_3^2$$

where

$$k_{buoyancy} = \rho \cdot g \cdot \frac{\pi d^2}{4}$$

The equivalent mass (or rather, rotational inertia) is obtained by imposing a unit rotational acceleration $\ddot{\theta} = 1$ to the system and determining the moment required. Let $\ddot{x}_i$ denote the linear acceleration of mass $i$ corresponding to the rotational acceleration $\ddot{\theta} = 1$. Then the moment about the center will be

$$M = J_{eq} \ddot{\theta} =
\begin{align*}
&= m_1 \cdot \ddot{x}_1 \times l_1 + m_2 \cdot \ddot{x}_2 \times l_2 + m \cdot \ddot{x}_3 \sqrt{l_3^2 + l^2} \\
&= m_1 \cdot \ddot{\theta} l_1 \times l_1 + m_2 \cdot \ddot{\theta} l_2 \times l_2 + m \cdot \ddot{\theta} (l_3^2 + l^2) \\
&= J_{eq} = m_1 l_1^2 + m_2 l_2^2 + m (l_3^2 + l^2)
\end{align*}$$

where $l$ denotes the length of the rigid vertical bar.

Problem 1.43
One way to deal with it is the following.

$$\begin{align*}
1 + 2i &= \sqrt{5} \cdot e^{i \cdot 107i} \\
3 - 4i &= 5 \cdot e^{-0.927i}
\end{align*}$$

$$\Rightarrow \quad \frac{1 + 2i}{3 - 4i} = \frac{\sqrt{5} \cdot e^{i \cdot 107i}}{5 \cdot e^{-0.927i}} = \frac{\sqrt{5}}{5} \cdot e^{2.034i}$$

Problem 1.53
Let $x_1(t) = 3 \sin 30t$, $x_2(t) = 3 \sin 29t$. Then

$$x(t) = x_1(t) + x_2(t) =
\begin{align*}
&= 3 \sin(30t) + 3 \sin(29t) \\
&= 3 \cdot 2 \left( \sin \frac{30t + 29t}{2} \cos \frac{30t - 29t}{2} \right) \\
&= 6 \sin(29.5t) \cos(0.5t)
\end{align*}$$

The resulting oscillation is shown below.
The maximum amplitude of the combined oscillation will be \textit{approximately} 6. The \textit{exact} value can be determined by calculating the values of \( t \) for which \( \sin(29.5t) = 1 \) and are close to \( 2\pi + k, \) \( k \) being an integer. At those points, \( \cos(0.5t) = 1. \) Then calculate \( x(t) \) @ the values found for \( t. \) Note that there will always be at least one value of \( k \) for which the maximum amplitude is 6! This phenomenon (beat) is very common in mechanical vibrations.

\textbf{Problem 1.62}

The pulse can be expressed as

\[
x(t) = \begin{cases} 
A \sin \frac{2\pi t}{\tau}, & \text{if } 0 \leq t \leq \frac{\tau}{2} \\
0, & \text{if } \frac{\tau}{2} < t \leq \tau 
\end{cases}
\]

\[
a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2A}{\tau} \int_0^{\frac{\tau}{2}} \sin \frac{2\pi t}{\tau} dt = \frac{2A}{\pi} 
\]

\[
a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos(n\omega t) dt = \frac{2A}{\tau} \int_0^{\frac{\tau}{2}} \sin \frac{2\pi t}{\tau} \cos(n\omega t) dt = \frac{A}{\tau} \int_0^{\frac{\tau}{2}} \left( \sin[(1 + n)\omega t] + \sin[(1 - n)\omega t] \right) dt = \\
= \begin{cases} 
0, & \text{if } n \text{ is odd} \\
\frac{2A}{(n - 1)(n + 1)\pi}, & \text{if } n \text{ is even}
\end{cases}
\]

Similarly,

\[
b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin(n\omega t) dt = \frac{2A}{\tau} \int_0^{\frac{\tau}{2}} \sin \frac{2\pi t}{\tau} \cos(n\omega t) dt = \frac{A}{\tau} \int_0^{\frac{\tau}{2}} \left( \cos[(1 + n)\omega t] + \cos[(1 - n)\omega t] \right) dt = \\
= \frac{A}{\tau} \left( \frac{\sin[(1 - n)\omega t]}{(1 - n)\omega} - \frac{\sin[(1 + n)\omega t]}{(1 + n)\omega} \right) \bigg|_0^{\frac{\tau}{2}} = \begin{cases} 
\frac{\sqrt{2}}{2}, & \text{if } n = 1 \\
0, & \text{if } n = 2, 3, 4, \ldots
\end{cases}
\]

Therefore,

\[
x(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega t - \frac{2A}{\pi} \sum_{n=2, 4, 6, \ldots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}
\]
Problem 2.74
Let \( x(t) \) be measured from the static equilibrium position of the mass. The kinetic energy of the system will be

\[
E_k = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_\theta \dot{\theta}^2 = \frac{1}{2} \left( m + \frac{J_\theta}{r^2} \right) \dot{\theta}^2
\]

The potential energy of the system will be

\[
E_{pot} = \frac{1}{2} k y^2 = \frac{1}{2} k 16x^2
\]

where \( y = \theta (4r) = 4x \) = deflection of the spring. Deriving with respect to time yields

\[
m\ddot{x} + \frac{J_\theta}{r^2} \ddot{\theta} + 16kx = 0 \Rightarrow \omega_n = \sqrt{\frac{16kr^2}{mr^2 + J_\theta}}
\]

Problem 2.10
Equilibrium of the pulley requires

\[
K_{eq} x = 2T
\]

where \( x \) is the displacement of the car, and \( T \) is the tension in the wire. If \( \delta_x \) is the total stretch of the wire, \( \delta_x = 2x \). But also, \( \delta_x = \frac{T}{K_{wire}} = \frac{T}{EA} \Rightarrow T = 2 \frac{EA}{L_1 + L_2} x \)

Therefore, \( 2T = 4 \frac{EA}{L_1 + L_2} x = K_{eq} x \Rightarrow K_{eq} = \frac{4EA}{L_1 + L_2} \)

Furthermore, the component of weight in the \( x \) direction will be \( mg \sin \theta = 2T_{st} = K_{eq} x \). Therefore,

\[
m = \frac{K_{eq} x_{st}}{g \sin \theta}
\]

and

\[
\omega_n = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{4EA}{m(L_1 + L_2)}}
\]