Handout on Yield Curves:

Adam B. Ashcraft

Objects:

Maturity: the maturity of a bond is the length of time over which payments are made to the holder. For example, Treasury bills are government bonds with a maturity of one year or less, while Treasury notes have a maturity between one and ten years, and Treasury bonds have a maturity of ten years or more.

Face value: the face value of a bond is a single payment made to the holder when the bond matures (at the end of the holding period).

Coupon Payments: coupon payments are interest payments made to the holder every year until the bond matures.

Coupon Rate: the coupon rate is the ratio of the regular coupon payment to face value. For example a 20% coupon rate on a $1000 bond implies the bond provides the holder $200 a year in interest payments. Note this rate is generally fixed.

Gross versus Net interest rates: consider borrowing $100 for one year and paying back $105 next year. The gross interest rate is the ratio of interest to principal, or 1.05 while the net interest is simply the gross interest rate less one, 0.05.

Discount bond (also zero-coupon) a discount bond has no coupon payments and only pays the face value at maturity. For example, a three-year zero-coupon bond with a face value of $1000 pays nothing over the next two years and $1000 in three years.

Yield to Maturity (YTM): the net interest rate which ensures that the current bond price is the same as the present discounted value of cash flows. For example, the YTM on a two-year discount bond with a face value of $100 and priced at $90 today would be the solution to the following:

$$\$90 = \$100/(1+i)^2$$

$$YTM = sqrt(100/90)-1 = 0.0541$$

One should think of the YTM as the average interest rate paid out over the maturity of the bond. It is a measure of return to an investor. Similarly check that a three-year zero-coupon bond with $100 face value also priced at $90 has a YTM of 0.0357, calculated as $(100/90)^{1/3}-1$. Do NOT use log approximations when calculating the yield to maturity.

Notation:

P_{mt} is the price of a discount bond of maturity m years at time t.

i_{mt} is the nominal interest rate for a maturity of m years at time t. In other words, it is the price of one dollar m years from time t at time t. Obviously when m > 1 this interest rate compounds future one year rates. For example $(1+i_{2t})= (1+i_{1t})(1+i_{1t+1})$. To simplify notation, it is assumed that the time index for future periods implies that this interest rate is expected, as the actual one-year interest rate at time t+1 is unknown at time t. Finally note that for discount bonds, the YTM is simply $(1+i_{mt})^{1/m}-1$. This notation is slightly different from the book, which can be confusing.

Bond Prices as Present Values:

Consider two discount bonds with a face value of $100, differing only in that one matures in 1 year, the other in 2 years.

The value of each of these bonds is simply the present discounted value of cash flows. If we assume everyone in the market values these cash flows identically, then one can claim that the market price for each of these bonds is simply the present value of cash flows. This claim is illustrated below:

$P_{1t} = \$100/(1+i_{1t})$
$P_{2t} = \$100/[(1+i_{1t})*(1+i_{1t+1})]$

These are the fundamental pricing relations for discount bonds. One could construct the price for a discount bond of any maturity in this manner. These equations are motivated in a different manner below in the section on arbitrage.

Arbitrage:

Compare two investment strategies. The first is to purchase the one-year discount bond, while the second is to purchase the two-year discount bond and sell it next year.

The purchase of the one-year discount bond costs us $\$P_{1t}$ today and returns \$100 next year, for a nominal gross return of $1+i_{1t}$, calculated as $\$100/P_{1t}$.

The purchase of the two-year discount bond costs us $\$P_{2t}$ today and returns $\$P_{1t+1}$ next year, as a two-year discount bond today becomes a one-year discount bond one year from now, for a nominal gross return of $\$P_{1t+1}/\$P_{2t}$.

Since neither of these investment strategies is subject to default risk, it seems reasonable to claim that they should yield the same return to investors. If the yield to maturity on the one-year discounts was higher, everyone would buy one-year bonds and sell two-year bonds until prices were the same. As a result of arbitrage, we have the following identity:

$1+i_{1t} = \$P_{1t+1}/\$P_{2t}$, or equivalently,
$\$P_{2t} = \$P_{1t+1}/(1+i_{1t})$.

The price of a two-year discount is simply the present discounted value of the expected price of a one-year discount, one year in the future. But we know the formula for the price of a two-year discount one year from now:

$\$P_{1t+1} = \$100/(1+i_{1t+1})$.

Substitute this equation into the arbitrage relationship above to finally arrive at the final equation:

$\$P_{2t} = \$100/[(1+1_{1t})(1+i_{1t+1})]$.

This implies that arbitrage between discount bonds of different maturities implies that the price of two-year discount bonds is simply the present discounted value of the payment in two years.

The Yield Curve:

The YTM on a bond having a maturity of m years is the net interest rate that makes the bond price equal to the present discounted value of future payments on the bond, denoted by ytm_{mt}. Obviously the YTM on a one-year discount is simply the nominal interest rate.

The yield curve is a function which maps the space of maturities into the space of net interest rates.

Note that since the YTM is the interest rate which equalizes price and present discounted value, one can write the following equality:

$$\$100/(1+ytm_{mt})^m = P_{mt} = \$100/[(1+i_{1t})(1+i_{1t+1})\ldots(1+i_{1t+m-1})]$$

Rearranging and taking logs results in an intuitive expression for the yield to maturity:

$$ytm_{mt} = (i_{1t}+i_{1t+1}+\ldots+i_{1t+m-1})/m$$

This implies that the yield to maturity is simple average of expected one-year interest rates over the life of the bond. From this simple equation one can interpret the slope of the yield curve in the following manner:

An upward slope to the yield curve implies that one-year interest rates are expected to increase in the future.

A downward slope to the yield curve implies that one-year interest rates are expected to fall in the future.

An example makes this clear. In January 1993, $i_{1t} = 3.5\%$ and $ytm_{2t} = 4.4\%$ so $i_{1t+1} = 2*ytm_{2t}-i_{1t} = 5.3\%$, 1.8% above the January 1993 net rate.