14.06 Lecture Notes
Intermediate Macroeconomics

George-Marios Angeletos
MIT Department of Economics

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Chapter 3

The Neoclassical Growth Model

- In the Solow model, agents in the economy (or the dictator) follow a simplistic linear rule for consumption and investment. In the Ramsey model, agents (or the dictator) choose consumption and investment optimally so as to maximize their individual utility (or social welfare).

3.1 The Social Planner

- In this section, we start the analysis of the neoclassical growth model by considering the optimal plan of a benevolent social planner, who chooses the static and intertemporal allocation of resources in the economy so as to maximize social welfare. We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

- Together with consumption and saving, we also endogenize labor supply.

3.1.1 Preferences

- Preferences are defined over streams of consumption and leisure $\{x_t\}_{t=0}^{\infty}$, where $x_t = (c_t, z_t)$, and are represented by a utility function $\mathcal{U} : \mathbb{X}^{\infty} \to \mathbb{R}$, where $\mathbb{X}$ is the domain of x_t, such that

$$\mathcal{U}\left(\{x_t\}_{t=0}^{\infty}\right) = \mathcal{U}\left(x_0, x_1, ...\right)$$

- We say that preferences are *recursive* if there is a function $W : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ such that, for all $\{x_t\}_{t=0}^{\infty}$,

$$\mathcal{U}\left(x_0, x_1, ...\right) = W\left[x_0, \mathcal{U}\left(x_1, x_2, ...\right)\right]$$

  We can then represent preferences as follows: A consumption-leisure stream $\{x_t\}_{t=0}^{\infty}$ induces a utility stream $\{\mathcal{U}_t\}_{t=0}^{\infty}$ according to the recursion

$$\mathcal{U}_t = W(x_t, \mathcal{U}_{t+1}).$$

  That is, utility in period t is given as a function of consumption in period t and utility in period $t+1$. W is called a *utility aggregator*. Finally, note that recursive preferences, as defined above, are both time-consistent and stationary.

- We say that preferences are *additively separable* if there are functions $\upsilon_t : \mathbb{X} \to \mathbb{R}$ such that

$$\mathcal{U}\left(\{x_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \upsilon_t(x_t).$$

  We then interpret $\upsilon_t(x_t)$ as the utility enjoyed in period 0 from consumption in period $t+1$.

- Throughout our analysis, we will assume that preferences are both recursive and additively separable. In other words, we impose that the utility aggregator W is linear in

u_{t+1} : There is a function $U : \mathbb{R} \to \mathbb{R}$ and a scalar $\beta \in \mathbb{R}$ such that $W(x,u) = U(x) + \beta u$. We can thus represent preferences in recursive form as

$$\mathcal{U}_t = U(x_t) + \beta \mathcal{U}_{t+1}.$$

Alternatively,

$$\mathcal{U}_t = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{t+\tau})$$

- β is called the *discount factor*. For preferences to be well defined (that is, for the infinite sum to converge) we need $\beta \in (-1,+1)$. Monotonicity of preferences imposes $\beta > 0$. Therefore, we restrict $\beta \in (0,1)$. The discount rate is given by ρ such that $\beta = 1/(1+\rho)$.

- U is sometimes called the per-period felicity or utility function. We let $\overline{z} > 0$ denote the maximal amount of time per period. We accordingly let $\mathbb{X} = \mathbb{R}_+ \times [0, \overline{z}]$. We finally impose that U is *neoclassical*, in that it satisfies the following properties:

  1. U is continuous and (although not always necessary) twice differentiable.

  2. U is strictly increasing and strictly concave:

  $$\begin{aligned} U_c(c,z) &> 0 > U_{cc}(c,z) \\ U_z(c,z) &> 0 > U_{zz}(c,z) \\ U_{cz}^2 &< U_{cc}U_{zz} \end{aligned}$$

  3. U satisfies the Inada conditions

  $$\begin{aligned} \lim_{c \to 0} U_c &= \infty \quad \text{and} \quad \lim_{c \to \infty} U_c = 0. \\ \lim_{z \to 0} U_z &= \infty \quad \text{and} \quad \lim_{z \to \overline{z}} U_z = 0. \end{aligned}$$

3.1.2 Technology and the Resource Constraint

- We abstract from population growth and exogenous technological change.

- The time constraint is given by

$$z_t + l_t \leq \overline{z}.$$

  We usually normalize $\overline{z} = 1$ and thus interpret z_t and l_t as the fraction of time that is devoted to leisure and production, respectively.

- The resource constraint is given by

$$c_t + i_t \leq y_t$$

- Let $F(K, L)$ be a neoclassical technology and let $f(\kappa) = F(\kappa, 1)$ be the intensive form of F. Output in the economy is given by

$$y_t = F(k_t, l_t) = l_t f(\kappa_t),$$

  where

$$\kappa_t = \frac{k_t}{l_t}$$

  is the capital-labor ratio.

- Capital accumulates according to

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

  (Alternatively, interpret l as effective labor and δ as the effective depreciation rate.)

- Finally, we impose the following natural non-negativitly constraints:

$$c_t \geq 0, \quad z_t \geq 0, \quad l_t \geq 0, \quad k_t \geq 0.$$

- Combining the above, we can rewrite the *resource constraint* as

$$c_t + k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t,$$

and the time constraint as

$$z_t = 1 - l_t,$$

with

$$c_t \geq 0, \quad l_t \in [0,1], \quad k_t \geq 0.$$

3.1.3 The Ramsey Problem

- The social planner chooses a plan $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ so as to maximize utility subject to the resource constraint of the economy, taking initial k_0 as given:

$$\max \mathcal{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

$$c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$

$$c_t \geq 0, \quad l_t \in [0,1], \quad k_{t+1} \geq 0., \quad \forall t \geq 0,$$

$$k_0 > 0 \;\; given.$$

3.1.4 Optimal Control

- Let μ_t denote the Lagrange multiplier for the resource constraint. The Lagrangian of the social planner's problem is

$$\mathcal{L}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t) + \sum_{t=0}^{\infty} \mu_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right]$$

- Define $\lambda_t \equiv \beta^t \mu_t$ and

$$\begin{aligned} H_t &\equiv H(k_t, k_{t+1}, c_t, l_t, \lambda_t) \equiv \\ &\equiv U(c_t, 1-l_t) + \lambda_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right] \end{aligned}$$

  H is called the *Hamiltonian* of the problem.

- We can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L}_0 &= \sum_{t=0}^{\infty} \beta^t \left\{U(c_t, 1-l_t) + \lambda_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right]\right\} = \\ &= \sum_{t=0}^{\infty} \beta^t H_t \end{aligned}$$

  or, in recursive form

$$\mathcal{L}_t = H_t + \beta \mathcal{L}_{t+1}.$$

- Given k_t, c_t and l_t enter only the period t utility and resource constraint; (c_t, l_t) thus appears only in H_t. Similarly, k_t,enter only the period t and $t+1$ utility and resource constraints; they thus appear only in H_t and H_{t+1}. Therefore,

Lemma 11 *If* $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ *is the optimum and* $\{\lambda_t\}_{t=0}^{\infty}$ *the associated multipliers, then*

$$(c_t, l_t) = \arg\max_{c,l} \overbrace{H(k_t, k_{t+1}, c, l, \lambda_t)}^{H_t}$$

taking (k_t, k_{t+1}) *as given, and*

$$k_{t+1} = \arg\max_{k'} \overbrace{H(k_t, k', c_t, l_t, \lambda_t) + \beta H(k', k_{t+2}, c_{t+1}, l_{t+1}, \lambda_{t+1})}^{H_t + \beta\ H_{t+1}}$$

taking (k_t, k_{t+2}) *as given.*

Equivalently,

$$\begin{aligned}(c_t, l_t, k_{t+1}, c_{t+1}, l_{t+1},) &= \arg\max_{c,l,k',c',l'} [U(c,l) + \beta U(c', l')] \\ s.t. \quad c + k' &\leq (1-\delta)k_t + F(k_t, l) \\ c' + k_{t+2} &\leq (1-\delta)k' + F(k', l')\end{aligned}$$

taking (k_t, k_{t+2}) *as given.*

- We henceforth assume an interior solution. As long as $k_t > 0$, interior solution is indeed ensured by the Inada conditions on F and U.

- The FOC with respect to c_t gives

$$\frac{\partial \mathcal{L}_0}{\partial c_t} = \beta^t \frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow$$
$$\frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow$$
$$U_c(c_t, z_t) = \lambda_t$$

The FOC with respect to l_t gives

$$\frac{\partial \mathcal{L}_0}{\partial l_t} = \beta^t \frac{\partial H_t}{\partial l_t} = 0 \Leftrightarrow$$
$$\frac{\partial H_t}{\partial l_t} = 0 \Leftrightarrow$$
$$U_z(c_t, z_t) = \lambda_t F_L(k_t, l_t)$$

Finally, the FOC with respect to k_{t+1} gives

$$\frac{\partial \mathcal{L}_0}{\partial k_{t+1}} = \beta^t \left[\frac{\partial H_t}{\partial k_{t+1}} + \beta \frac{\partial H_{t+1}}{\partial k_{t+1}} \right] = 0 \Leftrightarrow$$
$$-\lambda_t + \beta \frac{\partial H_{t+1}}{\partial k_{t+1}} = 0 \Leftrightarrow$$
$$\lambda_t = \beta \left[1 - \delta + F_K(k_{t+1}, l_{t+1}) \right] \lambda_{t+1}$$

- Combining the above, we get

$$\frac{U_z(c_t, z_t)}{U_c(c_t, z_t)} = F_L(k_t, l_t)$$

and

$$\frac{U_c(c_t, z_t)}{\beta U_c(c_{t+1}, z_{t+1})} = 1 - \delta + F_K(k_{t+1}, l_{t+1}).$$

- Both conditions impose equality between marginal rates of substitution and marginal rate of transformation. The first condition means that the marginal rate of substitution between consumption and leisure equals the marginal product of labor. The second condition means that the marginal rate of intertemporal substitution in consumption equals the marginal capital of capital net of depreciation (plus one). This last condition is called the *Euler condition*.

- The *envelope condition* for the Pareto problem is

$$\frac{\partial(\max \mathcal{U}_0)}{\partial k_0} = \frac{\partial \mathcal{L}_0}{\partial k_0} = \lambda_0 = U_c(c_0, z_0).$$

More generally,

$$\lambda_t = U_c(c_t, l_t)$$

represents the marginal utility of capital in period t and will equal the slope of the value function at $k = k_t$ in the dynamic-programming representation of the problem.

- Suppose for a moment that the horizon was finite, $T < \infty$. Then, the Lagrangian would be

$$\mathcal{L}_0 = \sum_{t=0}^{T} \beta^t H_t$$

and the Kuhn-Tucker condition with respect to k_{T+1} would give

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} = \beta^T \frac{\partial H_T}{\partial k_{T+1}} \geq 0 \quad \text{and} \quad k_{T+1} \geq 0, \text{ with complementary slackness;}$$

equivalently

$$\mu_T = \beta^T \lambda_T \geq 0 \text{ and } k_{T+1} \geq 0, \quad \text{with } \beta^T \lambda_T k_{T+1} = 0.$$

The latter means that either $k_{T+1} = 0$, or otherwise it better be that the shadow value of k_{T+1} is zero. When the horizon is infinite, the terminal condition $\beta^T \lambda_T k_{T+1} = 0$ is replaced by the transversality condition

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0.$$

Equivalently, using $\lambda_t = U_c(c_t, z_t)$, we write the transversality condition as

$$\lim_{t\to\infty} \beta^t U_c(c_t, z_t) k_{t+1} = 0.$$

The above means that, as time passes, the (discounted) shadow value of capital converges to zero.

- We conclude:

Proposition 12 *The plan* $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ *is a solution to the social planner's problem if and only if*

$$\frac{U_z(c_t, z_t)}{U_c(c_t, z_t)} = F_L(k_t, l_t), \tag{3.1}$$

$$\frac{U_c(c_t, z_t)}{\beta U_c(c_{t+1}, z_{t+1})} = 1 - \delta + F_K(k_{t+1}, l_{t+1}), \tag{3.2}$$

$$k_{t+1} = F(k_t, l_t) + (1-\delta)k_t - c_t, \tag{3.3}$$

for all $t \geq 0$, *and*

$$k_0 > 0 \ \text{ given, } \quad \text{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t, z_t) k_{t+1} = 0. \tag{3.4}$$

- *Remark:* We proved necessity of (3.1) and (3.2) essentially by a perturbation argument, and (3.3) is trivial. We did not prove necessity of (3.4), neither sufficiency of this set of conditions. One can prove both necessity and sufficiency using optimal-control techniques. Alternatively, we can use dynamic programming; the proof of the necessity and sufficiency of the Euler and transversality conditions is provided in Stokey and Lucas.

- Note that the (3.1) can be solved for $l_t = l(c_t, k_t)$, which we can then substitute into (3.2) and (3.3). We are then left with a system of two difference equations in two variables, namely c_t and k_t. The intitial condition and the transversality condition then give the boundary conditions for this system.

3.1.5 Dynamic Programing

Consider again the social planner's problem.

For any $k > 0$, define

$$V(k) \equiv \max \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

subject to

$$c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$

$$c_t, l_t, (1-l_t), k_{t+1} \geq 0, \quad \forall t \geq 0,$$

$$k_0 = k \ \ given.$$

V is called the Value Function.

- Define $\overline{k}$ by the unique solution to

$$\overline{k} = (1-\delta)\overline{k} + F(\overline{k}, 1)$$

and note that $\overline{k}$ represents an upper bound on the level of capital that can be sustained in any steady state. Without serious loss of generality, we will henceforth restrict $k_t \in [0, \overline{k}]$.

- Let B be the set of continuous and bounded functions $v : [0, \overline{k}] \to \mathbb{R}$ and consider the mapping $\mathcal{T} : B \to B$ defined as follows:

$$\begin{aligned} \mathcal{T}v(k) &= \max U(c, 1-l) + \beta v(k') \\ s.t. \quad & c + k' \leq (1-\delta)k + F(k,l) \\ k' \in [0, \overline{k}], & \quad c \in [0, F(k,1)], \quad l \in [0,1]. \end{aligned}$$

The conditions we have imposed on U and F imply that $\mathcal{T}$ is a contraction mapping. It follows that $\mathcal{T}$ has a unique fixed point $V = \mathcal{T}V$ and this fixed point gives the solution to the planner's problem:

Proposition 13 *There is a unique V that solves the Bellman equation*

$$\begin{aligned} V(k) &= \max U(c, 1-l) + \beta V(k') \\ s.t. \quad & c + k' \leq (1-\delta)k + F(k,l) \\ k' \in [0, \overline{k}], & \quad c \in [0, F(k,1)], \quad l \in [0,1]. \end{aligned}$$

V is continuous, differentiable, and strictly concave. $V(k_0)$ gives the solution for the social planner's problem.

Proposition 14 *Let*

$$[c(k), l(k), G(k)] = \arg\max\{...\}.$$

$c(k), l(k), G(k)$ are continuous; $c(k)$ and $G(k)$ are increasing. The plan $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ is optimal if and only if it satisfies

$$\begin{aligned} c_t &= c(k_t) \\ l_t &= l(k_t) \\ k_{t+1} &= G(k_t) \end{aligned}$$

with k_0 historically given.

- *Remark:* The proofs of the above propositions, as well as the proof of the necessity and sufficiency of the Euler and transversality conditions, are provided in Stokey and Lucas. Because of time constraints, I will skip these proofs and concentrate on the characterization of the optimal plan.

- The Lagrangian for the DP problem is

$$\mathcal{L} = U(c, 1-l) + \beta V(k') + \lambda[(1-\delta)k + F(k,l) - k' - c]$$

The FOCs with respect to c, l and k' give

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= 0 \Leftrightarrow U_c(c,z) = \lambda \\ \frac{\partial \mathcal{L}}{\partial l} &= 0 \Leftrightarrow U_z(c,z) = \lambda F_L(k,l) \\ \frac{\partial \mathcal{L}}{\partial k'} &= 0 \Leftrightarrow \lambda = \beta V_k(k') \end{aligned}$$

The Envelope condition is

$$V_k(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda[1 - \delta + F_K(k,l)]$$

- Combining, we conclude

$$\frac{U_z(c_t, l_t)}{U_c(c_t, l_t)} = F_l(k_t, l_t)$$

and

$$\frac{U_c(c_t, l_t)}{U_c(c_{t+1}, l_{t+1})} = \beta \left[1 - \delta + F_K(k_{t+1}, l_{t+1})\right],$$

which are the same conditions we had derived with optimal control. Finally, note that we can state the Euler condition alternatively as

$$\frac{V_k(k_t)}{V_k(k_{t+1})} = \beta[1 - \delta + F_K(k_{t+1}, l_{t+1})].$$

3.2 Decentralized Competitive Equilibrium

3.2.1 Households

- Households are indexed by $j \in [0, 1]$. There is one person per household and no population growth.

- The preferences of household j are given by

$$\mathcal{U}_0^j = \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$

In recursive form,

$$\mathcal{U}_t^j = U(c_t^j, z_t^j) + \beta \mathcal{U}_{t+1}^j$$

- The time constraint for household j can be written as

$$z_t^j = 1 - l_t^j.$$

- The budget constraint of household j is given by

$$c_t^j + i_t^j + x_t^j \leq y_t^j = r_t k_t^j + R_t b_t^j + w_t l_t^j + \alpha^j \Pi_t,$$

where r_t denotes the rental rate of capital, w_t denotes the wage rate, R_t denotes the interest rate on risk-free bonds. Household j accumulates capital according to

$$k_{t+1}^j = (1-\delta)k_t^j + i_t^j$$

and bonds according to

$$b_{t+1}^j = b_t^j + x_t^j$$

In equilibrium, firm profits are zero, because of CRS. It follows that $\Pi_t = 0$ and we can rewrite the household budget as

$$c_t^j + k_{t+1}^j + b_{t+1}^j \leq (1-\delta+r_t)k_t^j + (1+R_t)b_t^j + w_t l_t^j.$$

- The natural non-negativity constraint

$$k_{t+1}^j \geq 0$$

is imposed on capital holdings, but no short-sale constraint is imposed on bond holdings. That is, household can either lend or borrow in risk-free bonds. We only impose the following *natural borrowing limit*

$$-(1+R_{t+1})b_{t+1}^j \leq (1-\delta+r_{t+1})k_{t+1}^j + \sum_{\tau=t+1}^{\infty} \frac{q_\tau}{q_{t+1}} w_\tau.$$

where

$$q_t \equiv \frac{1}{(1+R_0)(1+R_1)...(1+R_t)} = (1+R_t)q_{t+1}.$$

This constraint simply requires that the net debt position of the household does not exceed the present value of the labor income he can attain by working all time.

- Note that simple arbitrage between bonds and capital implies that, in any equilibrium,

$$R_t = r_t - \delta.$$

That is, the interest rate on riskless bonds must equal the rental rate of capital net of depreciation. If $R_t < r_t - \delta$, all individuals would like to short-sell bonds (up to their borrowing constraint) and invest into capital. If $R_t > r_t - \delta$, capital would be dominated by bonds, and nobody in the economy would invest in capital. In the first case, there would be excess supply for bonds in the aggregate. In the second case, there would be excess demand for bonds and no investment in the aggregate. In equilibrium, R_t and r_t must adjust so that $R_t = r_t - \delta$.

- Provided that $R_t = r_t - \delta$, the household is indifferent between bonds and capital. The "portfolio" choice between k_t^j and b_t^j is thus indeterminate. What is pinned down is only the total asset position, $a_t^j = b_t^j + k_t^j$. The budget constraint then reduces to

$$c_t^j + a_{t+1}^j \leq (1 + R_t)a_t^j + w_t l_t^j,$$

and the natural borrowing constraint then becomes

$$a_{t+1}^j \geq \underline{a}_{t+1},$$

where

$$\underline{a}_{t+1} \equiv -\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau w_\tau$$

- We assume that $\{R_t, w_t\}_{t=0}^{\infty}$ satisfies

$$\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau w_\tau < M < \infty,$$

for all t, so that $\underline{a}_t$ is bounded away from $-\infty$. Note in particular that if $\sum_{\tau=t+1}^{\infty} q_\tau w_\tau$ was infinite at any t, the agent could attain infinite consumption in every period $\tau \geq t+1$.

- Given a price sequence $\{R_t, w_t\}_{t=0}^{\infty}$, household j chooses a plan $\{c_t^j, l_t^j, k_{t+1}^j\}_{t=0}^{\infty}$ so as to maximize lifetime utility subject to its budget constraints

$$\max \ \mathcal{U}_0^j = \sum_{t=0}^{\infty} \beta^t U(c_t^j, 1 - l_t^j)$$
$$s.t. \ c_t^j + a_{t+1}^j \leq (1 + R_t) a_t^j + w_t l_t^j$$
$$c_t^j \geq 0, \quad l_t^j \in [0, 1], \quad a_{t+1}^j \geq \underline{a}_{t+1}$$

- Let $\mu_t^j = \beta^t \lambda_t^j$ be the Lagrange multiplier for the budget constraint, we can write the Lagrangian as

$$\begin{aligned} \mathcal{L}_0^j &= \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t^j, 1 - l_t^j) + \lambda_t^j \left[(1 + R_t) a_t^j + w_t l_t^j - a_{t+1}^j - c_t^j \right] \right\} \\ &= \sum_{t=0}^{\infty} \beta^t H_t^j \end{aligned}$$

where

$$H_t^j = U(c_t^j, 1 - l_t^j) + \lambda_t^j \left[(1 + R_t) a_t^j + w_t l_t^j - a_{t+1}^j - c_t^j \right]$$

- The FOC with respect to c_t^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial c_t^j} = \beta^t \frac{\partial H_t^j}{\partial c_t^j} = 0 \Leftrightarrow$$
$$U_c(c_t^j, z_t^j) = \lambda_t^j$$

The FOC with respect to l_t^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial l_t^j} = \beta^t \frac{\partial H_t^j}{\partial l_t^j} = 0 \Leftrightarrow$$
$$U_z(c_t^j, z_t^j) = \lambda_t^j w_t$$

Combining, we get

$$\frac{U_z(c_t^j, z_t^j)}{U_c(c_t^j, z_t^j)} = w_t.$$

That is, households equate their marginal rate of substitution between consumption and leisure with the (common) wage rate.

- The Kuhn-Tucker condition with respect to a_{t+1}^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial a_{t+1}^j} = \beta^t \left[\frac{\partial H_t^j}{\partial a_{t+1}^j} + \beta \frac{\partial H_{t+1}^j}{\partial a_{t+1}^j} \right] \leq 0 \Leftrightarrow$$
$$\lambda_t^j \geq \beta \left[1 + R_t\right] \lambda_{t+1}^j,$$

with equality whenever $a_{t+1}^j > \underline{a}_{t+1}$. That is, the complementary slackness condition is

$$\left[\lambda_t^j - \beta \left[1 + R_t\right] \lambda_{t+1}^j\right] \left[a_{t+1}^j - \underline{a}_{t+1}\right] = 0.$$

- Finally, if time was finite, the terminal condition would be

$$\mu_T^j \geq 0, \quad a_{T+1}^j \geq \underline{a}_{T+1}, \quad \mu_T^j \left[a_{T+1}^j - \underline{a}_{T+1}\right] = 0,$$

where $\mu_t^j \equiv \beta^t \lambda_t^j$. Now that time is infinite, the transversality condition is

$$\lim_{t \to 0} \beta^t \lambda_t^j \left[a_{t+1}^j - \underline{a}_{t+1}\right] = 0.$$

- Using $\lambda_t^j = U_c(c_t^j, z_t^j)$, we can restate the Euler condition as

$$U_c(c_t^j, z_t^j) \geq \beta[1 + R_t]U_c(c_{t+1}^j, z_{t+1}^j),$$

with equality whenever $a_{t+1}^j > \underline{a}_{t+1}$. That is, as long as the borrowing constraint does not bind, households equate their marginal rate of intertemporal substitution with the (common) return on capital. On the other hand, if the borrowing constraint is binding, the marginal utility of consumption today may exceed the marginal benefit of savings: The household would like to borrow, but it's not capable of.

- For general borrowing limit $\underline{a}_t$, there is nothing to ensure that the Euler condition must be satisfied with equality. For example, if we had specified $\underline{a}_t = 0$, it likely the borrowing constraint will bind, especially if $\beta(1+R_t) < 1$ and w_t is low as compared to its long-run mean. But if $\underline{a}_t$ is the natural borrowing limit, and the utility satisfies the Inada condition $U_c \to \infty$ as $c \to 0$, then a simple argument ensures that the borrowing constraint can never bind: Suppose that $a_{t+1} = \underline{a}_{t+1}$. Then $c_\tau^j = z_\tau^j = 0$ for all $\tau \geq t$, implying $U_c(c_{t+1}^j, z_{t+1}^j) = \infty$ and therefore necessarily $U_c(c_t^j, z_t^j) < \beta[1 + R_t]U_c(c_t^j, z_t^j)$, unless $c_t^j = 0$ which would be optimal only if $a_t = \underline{a}_t$. Therefore, unless $a_0 = \underline{a}_0$ to start with, the borrowing which would contradict the Euler condition. Therefore, $a_t > \underline{a}_t$ at all dates, and the Euler condition is satisfied with equality:

- Moreover, if the borrowing constraint never binds, iterating $\lambda_t^j = \beta\left[1 + R_t\right]\lambda_{t+1}^j$ implies

$$\beta^t \lambda_t^j = q_t \lambda_0^j.$$

We can therefore rewrite the transversality as

$$\lim_{t\to\infty} \beta^t \lambda_t^j a_{t+1}^j = \lim_{t\to\infty} \beta^t \lambda_t^j \underline{a}_{t+1} = \lambda_0^j \lim_{t\to\infty} q_t \underline{a}_{t+1}$$

But note that

$$q_t \underline{a}_{t+1} = \sum_{\tau=t}^{\infty} q_\tau w_\tau$$

and $\sum_{\tau=0}^{\infty} q_\tau w_\tau < \infty$ implies $\lim_{t\to\infty} \sum_{\tau=t}^{\infty} q_\tau w_\tau = 0$. Therefore, the transversality condition reduces to

$$\lim_{t\to\infty} \beta^t \lambda_t^j a_{t+1}^j = 0$$

Equivalently,

$$\lim_{t\to\infty} \beta^t U_c(c_t^j, z_t^j) a_{t+1}^j = 0.$$

- It is useful to restate the household problem in a "static" format (that's essentially assuming complete Arrow-Debreu markets). As long as the borrowing constraint does not bind and the Inada conditions hold, we can rewrite the household problem as

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$

$$s.t. \; \sum_{t=0}^{\infty} q_t \cdot c_t^j + \sum_{t=0}^{\infty} q_t w_t \cdot z_t^j \leq \overline{x}$$

where

$$\overline{x} \equiv q_0(1 + R_0)a_0 + \sum_{t=0}^{\infty} q_t w_t < \infty.$$

- The constraint follows by integrating the per-period budgets for all $t \geq 0$ and is called the *intertemporal budget constraint*. We assume that $q_t > 0$, $w_t > 0$, and

$$\sum_{t=0}^{\infty} q_t < \infty \;\; \text{and} \;\; \sum_{t=0}^{\infty} q_t w_t < \infty,$$

which ensures that the set of feasible $\{c_t^j, z_t^j\}_{t=0}^{\infty}$ is compact. The FOCs give

$$\beta^t U_c(c_t^j, z_t^j) = \mu q_t,$$

$$\beta^t U_z(c_t^j, z_t^j) = \mu q_t w_t,$$

where $\mu > 0$ is Lagrange multiplier associated to the intertermporal budget. You can check that these conditions coincide with the one derived before.

- Finally, note that the objective is strictly concave and the constraint is linear. Therefore, the FOCs together with the transversality are both necessary and sufficient. We conclude:

Proposition 15 *Suppose the price sequence* $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ *satisfies* $R_t = r_t - \delta$ *for all* t, $\sum_{t=0}^{\infty} q_t < \infty$, *and* $\sum_{t=0}^{\infty} q_t w_t < \infty$,. *The plan* $\{c_t^j, l_t^j, a_t^j\}_{t=0}^{\infty}$ *solves the individual household's problem if and only if*

$$\frac{U_z(c_t^j, z_t^j)}{U_c(c_t^j, z_t^j)} = w_t,$$

$$\frac{U_c(c_t^j, z_t^j)}{\beta U_c(c_{t+1}^j, z_{t+1}^j)} = 1 + R_t,$$

$$c_t^j + a_{t+1}^j = (1 + R_t)a_t^j + w_t l_t^j,$$

for all $t \geq 0$, *and*

$$a_0^j > 0 \quad \text{given,} \quad \text{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t^j, z_t^j)a_{t+1}^j = 0.$$

Given $\{a_t^j\}_{t=1}^{\infty}$, *an optimal portfolio is any* $\{k_t^j, b_t^j\}_{t=1}^{\infty}$ *such that* $k_t^j \geq 0$ *and* $b_t^j = a_t^j - k_t^j$.

- *Remark:* For a more careful discussion on the necessity and sufficiency of the FOCs and the transversality condition, check Stokey and Lucas.

3.2.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seeks to maximize profits. The FOCs for an interior solution require

$$\begin{aligned} F_K(K_t^m, L_t^m) &= r_t. \\ F_L(K_t^m, L_t^m) &= w_t. \end{aligned}$$

  You can think of the first condition as the firm's demand for labor and the second condition as the firm's demand for capital.

- As we showed before in the Solow model, under CRS, an interior solution to the firms' problem to exist if and only if r_t and w_t imply the same K_t^m/L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$\begin{aligned} r_t &= f'(X_t) \\ w_t &= f(X_t) - f'(X_t)X_t \end{aligned}$$

  where $f(k) \equiv F(k, 1)$. Provided so, firm profits are zero

$$\Pi_t^m = 0$$

  and the FOCs reduce to

$$K_t^m = X_t L_t^m.$$

  That is, the FOCs pin down the capital labor ratio for each firm (K_t^m/L_t^m), but not the size of the firm (L_t^m). Moreover, because all firms have access to the same technology, they use exactly the same capital-labor ratio. (See our earlier analysis in the Solow model for more details.)

3.2.3 Market Clearing

- There is no exogenous aggregate supply of riskless bonds. Therefore, the *bond market* clears if and only if
$$0 = \int_0^{L_t} b_t^j dj.$$

- The *capital market* clears if and only if
$$\int_0^{M_t} K_t^m dm = \int_0^1 k_t^j dj$$
Equivalently,
$$\int_0^{M_t} K_t^m dm = k_t$$
where $k_t = K_t \equiv \int_0^1 k_t^j dj$ is the aggregate and per-head supply of capital in the economy.

- The *labor market*, on the other hand, clears if and only if
$$\int_0^{M_t} L_t^m dm = \int_0^{L_t} l_t^j dj$$
Equivalently,
$$\int_0^{M_t} L_t^m dm = l_t$$
where $l_t = L_t \equiv \int_0^{L_t} l_t^j dj$ is the aggregate and per-head supply of labor force in the economy.

3.2.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is quite natural:

Definition 16 *An equilibrium of the economy is an allocation* $\{(c_t^j, l_t^j, k_{t+1}^j, b_{t+1}^j)_{j\in[0,L_t]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=}^{\infty}$ *and a price path* $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ *such that* *(i) Given* $\{R_t, r_t, w_t\}_{t=0}^{\infty}$*, the path* $\{c_t^j, l_t^j, k_{t+1}^j, b_{t+1}^j\}$ *maximizes the utility of of household* j*, for every* j*.*

(ii) Given (r_t, w_t)*, the pair* (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t*.*

(iii) The bond, capital and labor markets clear in every period

- *Remark:* In the above definition we surpassed the distribution of firm profits (or the stock market). As we explained before in the Solow model, this is without any serious loss of generality because firm profits (and thus firm value) is zero.

3.2.5 General Equilibrium: Existence, Uniqueness, and Characterization

- In the Solow model, we had showed that the decentralized market economy and the centralized dictatorial economy were isomorphic. A similar result applies in the Ramsey model. The following proposition combines the first and second fundamental welfare theorems, as applied in the Ramsey model:

Proposition 17 *The set of competitive equilibrium allocations for the market economy coincide with the set of Pareto allocations for the social planner.*

- **Proof.** I will sketch the proof assuming that **(a)** in the market economy, $k_0^j + b_0^j$ is equal across all j; and **(b)** the social planner equates utility across agents. For the more general case, we need to extend the social planner's problem to allow for an unequal distribution of consumption and wealth across agents. The set of competitive equilibrium allocations coincides with the set of Pareto optimal allocations, each different competitive equilibrium allocation corresponding to a different system of Pareto weights in the utility of the social planner. I also surpass the details about the boundedness of prices. For a more careful analysis, see Stokey and Lucas.

a. We first consider how the solution to the social planner's problem can be implemented as a competitive equilibrium. The social planner's optimal plan is given by $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ such that

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t), \quad \forall t \geq 0,$$
$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta[1-\delta+F_K(k_{t+1}, l_{t+1})], \quad \forall t \geq 0,$$
$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$
$$k_0 > 0 \text{ given,} \quad \text{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)k_{t+1} = 0.$$

Choose a price path $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ such that

$$\begin{aligned} R_t &= r_t - \delta, \\ r_t &= F_K(k_t, l_t) = f'(\kappa_t), \\ w_t &= F_L(k_t, l_t) = f(\kappa_t) - f'(\kappa_t)\kappa_t, \end{aligned}$$

where $\kappa_t \equiv k_t/l_t$. Trivially, these prices ensure that the FOCs are satisfied for every household and every firm if we set $c_t^j = c_t$, $l_t^j = l_t$ and $K_t^m/L_t^m = k_t$ for all j and m. Next, we need to verify that the proposed allocation satisfies the budget constraints of each household. From the resource constraint of the economy,

$$c_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t.$$

From CRS and the FOCs for the firms,

$$F(k_t, l_t) = r_t k_t + w_t l_t.$$

Combining, we get

$$c_t + k_{t+1} = (1 - \delta + r_t)k_t + w_t l_t.$$

As long as $c_t^j = c_t$, $l_t^j = l_t$, and $a_t^j = k_t^j + b_t^j = k_t$ for all j, t, and $R_t = r_t - \delta$ for all t, it follows that

$$c_t^j + k_{t+1}^j + b_{t+1}^j = (1 - \delta + r_t)k_t^j + (1 + R_t)b_t^j + w_t l_t^j,$$

which proves that the budget constraint is satisfied for every j, t. Finally, it is trivial to that the proposed allocations clear the bond, capital, and labor markets.

b. We next consider the converse, how a competitive equilibrium coincides with the Pareto solution. Because agents have the same preferences, face the same prices, and are endowed with identical level of initial wealth, and because the solution to the individual's problem is essentially unique (where essentially means unique with respect to c_t^j, l_t^j, and $a_t^j = k_t^j + b_t^j$ but indeterminate with respect to the portfolio choice between k_t^j and b_t^j), every agent picks the same allocations: $c_t^j = c_t$, $l_t^j = l_t$ and $a_t^j = a_t$ for all j, t. By the FOCs to the individual's problem, it follows that $\{c_t, l_t, a_t\}_{t=0}^{\infty}$ satisfies

$$\frac{U_z(c_t, 1 - l_t)}{U_c(c_t, 1 - l_t)} = w_t, \quad \forall t \geq 0,$$
$$\frac{U_c(c_t, 1 - l_t)}{U_c(c_{t+1}, 1 - l_{t+1})} = \beta[1 - \delta + r_t], \quad \forall t \geq 0,$$
$$c_t + a_{t+1} = (1 - \delta + r_t)a_t + w_t l_t, \quad \forall t \geq 0,$$
$$a_0 > 0 \text{ given}, \quad \text{and} \quad \lim_{t \to \infty} \beta^t U_c(c_t, 1 - l_t)a_{t+1} = 0.$$

From the market clearing conditions for the capital and bond markets, the aggregate supply of bonds is zero and thus

$$a_t = k_t.$$

Next, by the FOCs for the firms,

$$\begin{aligned} r_t &= F_K(k_t, l_t) \\ w_t &= F_L(k_t, l_t) \end{aligned}$$

and by CRS

$$r_t k_t + w_t l_t = F(k_t, l_t)$$

Combining the above with the FOCs and the budget constraints gives

$$c_t + k_{t+1} = F(k_t, l_t) + (1 - \delta)k_t, \quad \forall t \geq 0,$$

which is simply the resource constraint of the economy. Finally, and $\lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)a_{t+1} = 0$ with $a_{t+1} = k_{t+1}$ implies the social planner's transversality condition, while $a_0 = k_0$ gives the initial condition. This concludes the proof that the competitive equilibrium coincides with the social planner's optimal plan. **QED**

- Following the above, we have:

Proposition 18 *An equilibrium always exists. The allocation of production across firms is indeterminate, and the portfolio choice of each household is also indeterminate, but the equilibrium is unique as regards prices, aggregate allocations, and the distribution of consumption, labor and wealth across households. If initial wealth $k_0^j + b_0^j$ is equal across all agent j, then $c_t^j = c_t$, $l_t^j = l_t$ and $k_t^j + b_t^j = k_t$ for all j. The equilibrium is then given by an allocation $\{c_t, l_t, k_t\}_{t=0}^\infty$ such that, for all $t \geq 0$,*

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t),$$

$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta[1 - \delta + F_K(k_{t+1}, l_{t+1})],$$

$$k_{t+1} = F(k_t, l_t) + (1-\delta)k_t - c_t,$$

and such that

$$k_0 > 0 \quad \textit{given,} \quad \textit{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)k_{t+1} = 0.$$

Finally, equilibrium prices are given by

$$\begin{aligned} R_t &= R(k_t) \equiv f'(k_t) - \delta, \\ r_t &= r(k_t) \equiv f'(k_t), \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t, \end{aligned}$$

where $R'(k) = r'(k) < 0 < w'(k)$.

Proof. The characterization of the equilibrium follows from our previous analysis. Existence and uniqueness of the equilibrium follow directly from existence and uniqueness of the social planner's optimum, given the coincidence of competitive and Pareto allocations. See Stokey and Lucas for more details. **QED** ■

3.3 Steady State and Transitional Dynamics

3.3.1 Steady State

- A steady state is a fixed point (c, l, k) of the dynamic system. A trivial steady state is at $c = l = k = 0$. We now consider interior steady states.

Proposition 19 *There exists a unique steady state* $(c^*, l^*, k^*) > 0$. *The steady-state values of the capital-labor ratio, the productivity of labor, the output-capital ratio, the consumption-capital ratio, the wage rate, the rental rate of capital, and the interest rate are all independent of the utility function* U *and are pinned down uniquely by the technology* F, *the depreciation* *rate δ, and the discount rate ρ. In particular, the capital-labor ratio $\kappa^* \equiv k^*/l^*$ equates the net-of-depreciation MPK with the discount rate,*

$$f'(\kappa^*) - \delta = \rho,$$

and is a decreasing function of $\rho + \delta$, where $\rho \equiv 1/\beta - 1$. Similarly,

$$R^* = \rho, \quad r^* = \rho + \delta,$$
$$w^* = F_L(\kappa^*, 1) = \frac{U_z(c^*, 1 - l^*)}{U_c(c^*, 1 - l^*)},$$
$$\frac{y^*}{l^*} = f(\kappa^*), \quad \frac{y^*}{k^*} = \phi(\kappa^*), \quad \frac{c^*}{k^*} = \frac{y^*}{k^*} - \delta,$$

where $f(\kappa) \equiv F(\kappa, 1)$ and $\phi(\kappa) \equiv f(\kappa)/\kappa$.

Proof. (c^*, l^*, k^*) must solve

$$\frac{U_z(c^*, 1 - l^*)}{U_c(c^*, 1 - l^*)} = F_L(k^*, l^*),$$
$$1 = \beta[1 - \delta + F_K(k^*, l^*)],$$
$$c^* = F(k^*, l^*) - \delta k^*,$$

Let $\kappa \equiv k/l$ denote the capital-labor ratio at the stead state. By CRS,

$$\begin{aligned} F(k,l) &= lf(\kappa) \\ F_K(k,l) &= f'(\kappa) \\ F_L(k,l) &= f(\kappa) - f'(\kappa)\kappa \\ \frac{F(k,l)}{k} &= \phi(\kappa) \end{aligned}$$

where $f(\kappa) \equiv F(\kappa, 1)$ and $\phi(\kappa) \equiv f(\kappa)/\kappa$. The Euler condition then reduces to

$$1 = \beta[1 - \delta + f'(\kappa^*)]$$

That is, the capital-labor ratio is pinned down uniquely by the equation of the MPK, net of depreciation, with the discount rate

$$f'(\kappa^*) - \delta = \rho$$

where $\rho \equiv 1/\beta - 1$ or, equivalently, $\beta \equiv 1/(1+\rho)$. The gross rental rate of capital and the net interest rate are thus

$$r^* = \rho + \delta \quad \text{and} \quad R^* = \rho,$$

while the wage rate is

$$w^* = F_{L:}(\kappa^*, 1)$$

The average product of labor and the average product of capital are given by

$$\frac{y^*}{l^*} = f(\kappa^*) \quad \text{and} \quad \frac{y^*}{k^*} = \phi(\kappa^*),$$

while, by the resource constraint, the consumption-capital ratio is given by

$$\frac{c^*}{k^*} = \phi(\kappa^*) - \delta = \frac{y^*}{k^*} - \delta.$$

The comparative statics are then trivial. **QED** ■

3.3.2 Transitional Dynamics

- Consider the condition that determined labor supply:

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t).$$

We can solve this for l_t as a function of contemporaneous consumption and capital:

$$l_t = l(c_t, k_t).$$

Substituting then into the Euler condition and the resource constraint, we conclude:

$$\begin{aligned} \frac{U_c(c_t, 1-l(c_t,k_t))}{U_c(c_t, 1-l(c_t,k_t))} &= \beta[1-\delta+F_K(k_{t+1}, l(c_{t+1},k_{t+1}))] \\ k_{t+1} &= F(k_t, l(c_t,k_t)) + (1-\delta)k_t - c_t \end{aligned}$$

This is a system of two first-order difference equation in c_t and k_t. Together with the initial condition (k_0 given) and the transversality condition, this system pins down the path of $\{c_t, k_t\}_{t=0}^{\infty}$.

3.4 The Neoclassical Growth Model with Exogenous Labor

3.4.1 Steady State and Transitional Dynamics

- Suppose that leisure is not valued, or that the labor supply is exogously fixed. Either way, let $l_t = 1$ for all t. Suppose further that preferences exhibit constant elasticity of intertemporal substitution:

$$U(c) = \frac{c^{1-1/\theta}-1}{1-1/\theta},$$

  $\theta > 0$ is reciprocal of the elasticity of the marginal utility of consumption and is called the elasticity of intertemporal substitution. Under these restrictions, the dynamics reduce to

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta[1+f'(k_{t+1})-\delta] = \beta[1+R_t],$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t.$$

- Finally, we know that the transversality condition is satisfied if and only if the path converges to the steady state, and we can also so that the capital stock converges monotonically to its steady state value. We conclude:

Proposition 20 *Suppose that labor is exogenously fixed and preferences exhibit CEIS. The path* $\{c_t, k_t\}_{t=0}^{\infty}$ *is the equilibrium path of the economy (and the solution to the social planner's problem) if and only if*

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+}) - \delta]\}^{\theta},$$

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t,$$

for all t*, with*

$$k_0 \text{ given and } \lim_{t\to\infty} k_t = k^*,$$

where k^* *is the steady state value of capital:*

$$f'(k^*) = \rho + \delta$$

For any initial $k_0 < k^*$ $(k_0 > k^*)$*, the capital stock* k_t *is increasing (respectively, decreasing) over time and converges to asymptotically to* k^**. Similarly, the rate of per-capita consumption growth* c_{t+1}/c_t *is positive and decreasing (respectively, negative and increasing) over time and converges monotonically to* 0.

Proof. The policy rule $k_{t+1} = G(k_t)$ is increasing, continuous, and intersects with the 45^{o} only at $k = 0$ and $k = k^*$. See Lucas and Stokey for the complete proof. The same argument as in the Solow model then implies that $\{k_t\}_{t=0}^{\infty}$ is monotonic and converges to k^*. The monotonicity and convergence of $\{c_{t+1}/c_t\}_{t=0}^{\infty}$ then follows immediately from the monotonicity and convergence of $\{k_t\}_{t=0}^{\infty}$ together with the fact that $f'(k)$ is decreasing. ■

- We will see these results also graphically in the phase diagram, below.

3.4.2 Continuous Time

- Taking logs of the Euler condition and approximating $\ln \beta = -\ln(1+\rho) \approx -\rho$ and $\ln[1-\delta+f'(k_t)] \approx f'(k_t)-\delta$, we can write the Euler condition as

$$\ln c_{t+1} - \ln c_t \approx \theta[f'(k_{t+1}) - \delta - \rho].$$

We can also rewrite the resource constraint as

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t.$$

- The approximation turns out to be exact when time is continuous:

Proposition 21 *Suppose that time is continuous. Like before, assume that labor is exogenously fixed and preferences exhibit CEIS. The path $\{c_t, k_t\}_{t\in\mathbb{R}_+}$ is the equilibrium path of the economy (and the solution to the social planner's problem) if and only if*

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho] = \theta[R_t - \rho],$$
$$\dot{k}_t = f(k_t) - \delta k_t - c_t,$$

for all t, with

$$k_0 \text{ given and } \lim_{t\to\infty} k_t = k^*,$$

where k^* is the steady state value of capital:

$$f'(k^*) = \rho + \delta$$

Proof. See Barro and Sala-i-Martin for details. ■

3.4.3 Phase Diagram (Figure 1)

- We can now use the phase diagram to describe the transitional dynamics of the economy. See **Figure 1.**

- The $\dot{k} = 0$ locus is given by (c, k) such that

$$\dot{k} = f(k) - \delta k - c = 0 \Leftrightarrow$$
$$c = f(k) - \delta k$$

  On the other hand, the $\dot{c} = 0$ locus is given by (c, k) such that

$$\dot{c} = c\theta[f'(k) - \delta - \rho] = 0 \Leftrightarrow$$
$$k = k^*$$

  Remark: Obviously, $c = 0$ also ensures $\dot{c} = 0$, but this corresponds to the trivial and unstable steady state $c = 0 = k$, so I will ignore it for the rest of the discussion.

- The steady state is simply the intersection of the two loci:

$$\dot{c} = \dot{k} = 0 \;\; \Leftrightarrow$$
$$\left\{ \begin{array}{c} k = k^* \equiv (f')^{-1}(\rho + \delta) \\ c = c^* \equiv f(k^*) - \delta k^* \end{array} \right\} \text{ or } \{c = k = 0\}$$

- The $\dot{c} = 0$ and $\dot{k} = 0$ loci are depicted in Figure 1. Note that the two loci partition the (c, k) space in four regions. We now examine what is the direction of change in c and k in each of these four regions.

- Consider first the direction of $\dot{c}$. If $0 < k < k^*$ $[k > k^*]$, then and only then $\dot{c} > 0$ $[\dot{c} < 0]$. That is, c increases [decreases] with time whenever (c, k) lies the left [right] of the $\dot{c} = 0$ locus. The direction of $\dot{c}$ is represented by the vertical arrows in Figure 1.

- Consider next the direction of $\dot{k}$. If $c < f(k) - \delta k$ [$c > f(k) - \delta k$], then and only then $\dot{k} > 0$ [$\dot{k} < 0$]. That is, k increases [decreases] with time whenever (c, k) lies below [above] the $\dot{k} = 0$ locus. The direction of $\dot{k}$ is represented by the horizontal arrows in Figure 1.

- We can now draw the time path of $\{k_t, c_t\}$ starting from any arbitrary (k_0, c_0), as in Figure 1. Note that there are only two such paths that go through the steady state. The one with positive slope represents the stable manifold or saddle path. The other corresponds to the unstable manifold.

- The equilibrium path of the economy for any initial k_0 is given by the stable manifold. That is, for any given k_0, the equilibrium c_0 is the one that puts the economy on the saddle path.

- To understand why the saddle path is the optimal path when the horizon is infinite, note the following:

  - Any c_0 that puts the economy *above* the saddle path leads to zero capital and zero consumption in finite time, thus violating the Euler condition at that time. Of course, if the horizon was finite, such a path would have been the equilibrium path. But with infinite horizon it is better to consume less and invest more in period 0, so as to never be forced to consume zero at finite time.

  - On the other hand, any c_0 that puts the economy *below* the saddle path leads to so much capital accumulation in the limit that the transversality condition is violated. Actually, in finite time the economy has cross the golden-rule and will henceforth become dynamically inefficient. One the economy reaches k_{gold}, where $f'(k_{gold}) - \delta = 0$, continuing on the path is dominated by an alternative feasible path, namely that of investing nothing in new capital and consuming $c = f(k_{gold}) - \delta k_{gold}$ thereafter. In other words, the economy is wasting too much resources in investment and it would better increase consumption.

- Let the function $c(k)$ represent the saddle path. In terms of dynamic programming, $c(k)$ is simply the optimal policy rule for consumption given capital k. Equivalently, the optimal policy rule for capital accumulation is given by

$$\dot{k} = f(k) - \delta k - c(k),$$

or in discrete time

$$k_{t+1} \approx G(k_t) \equiv f(k_t) + (1-\delta)k_t - c(k_t).$$

- Finally, note that, no matter what is the form of $U(c)$, you could also write the dynamics in terms of k and λ:

$$\begin{aligned} \frac{\dot{\lambda}_t}{\lambda_t} &= f'(k_t) - \delta - \rho \\ \dot{k}_t &= f(k_t) - \delta k_t - c(\lambda_t), \end{aligned}$$

where $c(\lambda)$ solves $U_c(c) = \lambda$, that is, $c(\lambda) \equiv U_c^{-1}(\lambda)$. Note that $U_{cc} < 0$ implies $c'(\lambda) < 0$. As an exercise, you can draw the phase diagram and analyze the dynamics in terms of k and λ.

3.5 Comparative Statics and Impulse Responses

3.5.1 Additive Endowment (Figure 2)

- Suppose that each household receives an endowment $e > 0$ from God. Then, the household budget is

  $$c_t^j + k_{t+1}^j = w_t + r_t k_t^j + (1-\delta)k_t^j + e$$

  Optimal consumption growth is thus given again by

  $$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + r_{t+1} - \delta]$$

  which together with $r_t = f'(k_t)$ implies

  $$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+1}) - \delta]\}^\theta$$

  On the other hand, adding up the budget across households gives the resource constraint of the economy

  $$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t + e$$

- We conclude that the phase diagram becomes

  $$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$
  $$\dot{k}_t = f(k_t) - \delta k_t - c_t + e.$$

- In the steady state, k^* is independent of e and c^* moves one to one with e.

- Consider a permanent increase in e by Δe. This leads to a parallel shift in the $\dot{k} = 0$ locus, but no change in the $\dot{c} = 0$ locus. If the economy was initially at the steady state, then k stays constant and c simply jumps by exactly e. On the other hand, if the economy was below the steady state, c will initially increase but by less that e, so that both the level and the rate of consumption growth will increase along the transition. See **Figure 2.**

3.5.2 Taxation and Redistribution (Figures 3 and 4)

- Suppose that labor and capital income are taxed at a flat tax rate $\tau \in (0,1)$. The government redistributes the proceeds from this tax uniformly across households. Let T_t be the transfer in period t. Then, the household budget is

$$c_t^j + k_{t+1}^j = (1-\tau)(w_t + r_t k_t^j) + (1-\delta)k_t^j + T_t,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + (1-\tau)r_{t+1} - \delta].$$

That is, the tax rate decreases the private return to investment. Combining with $r_t = f'(k_t)$ we infer

$$\frac{c_{t+1}}{c_t} = \left\{\beta[1 + (1-\tau)f'(k_{t+1}) - \delta]\right\}^{\theta}.$$

Adding up the budgets across household gives

$$c_t + k_{t+1} = (1-\tau)f(k_{t+1}) + (1-\delta)k_t + T_t$$

The government budget on the other hand is

$$T_t = \tau \int_j (w_t + r_t k_t^j) = \tau f(k_t)$$

Combining we get the resource constraint of the economy:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$$

Observe that, of course, the tax scheme does not appear in the resource constraint of the economy, for it is only redistributive and does not absorb resources.

- We conclude that the phase diagram becomes

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = f(k_t) - \delta k_t - c_t.$$

- In the steady state, k^* and therefore c^* are decreasing functions of τ.

A. Unanticipated Permanent Tax Cut

- Consider an unanticipated permanent tax cut that is enacted immediately. The $\dot{k} = 0$ locus does not change, but the $\dot{c} = 0$ locus shifts right. The saddle path thus shifts right. See **Figure 3.**

- A permanent tax cut leads to an immediate negative jump in consumption and an immediate positive jump in investment. Capital slowly increases and converges to a higher k^*. Consumption initially is lower, but increases over time, so soon it recovers and eventually converges to a higher c^*.

B. Anticipated Permanent Tax Cut

- Consider an permanent tax cut that is (credibly) announced at date 0 to be enacted at some date $\hat{t} > 0$. The difference from the previous exercise is that $\dot{c} = 0$ locus now does not change immediately. It remains the same for $t < \hat{t}$ and shifts right only for $t > \hat{t}$. Therefore, the dynamics of c and k will be dictated by the "old" phase diagram (the one corresponding to high τ) for $t < \hat{t}$ and by the "new" phase diagram (the one corresponding to low τ) for $t > \hat{t}$,

- At $t = \hat{t}$ and on, the economy must follow the saddle path corresponding to the new low τ, which will eventually take the economy to the new steady state. For $t < \hat{t}$, the economy must follow a path dictated by the old dynamics, but at $t = \widehat{t}$ the economy must exactly reach the new saddle path. If that were not the case, the consumption path would have to jump at date $\widehat{t}$, which would violate the Euler condition (and thus be suboptimal). Therefore, the equilibrium c_0 is such that, if the economy follows a path dictated by the old dynamics, it will reach the new saddle path exactly at $t = \widehat{t}$. See **Figure 4.**

- Following the announcement, consumption jumps down and continues to fall as long as the tax cut is not initiated. The economy is building up capital in anticipation of the tax cut. As soon as the tax cut is enacted, capital continues to increase, but consumption also starts to increase. The economy then slowly converges to the new higher steady state.

3.5.3 Productivity Shocks: A prelude to RBC (Figures 5 and 6)

- We now consider the effect of a shock in total factor productivity (TFP). The reaction of the economy in our deterministic framework is similar to the impulse responses we get in a stochastic Real Business Cycle (RBC) model. Note, however, that here we consider the case that labor supply is exogenously fixed. The reaction of the economy will be somewhat different with endogenous labor supply, whether we are in the deterministic or the stochastic case.

- Let output be given by

$$y_t = A_t f(k_t)$$

where A_t denotes TFP. Note that

$$\begin{aligned} r_t &= A_t f'(k_t) \\ w_t &= A_t[f(k_t) - f'(k_t)k_t] \end{aligned}$$

so that both the return to capital and the wage rate are proportional to TFP.

- We can then write the dynamics as

$$\frac{\dot{c}_t}{c_t} = \theta[A_t f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = A_t f(k_t) - \delta k_t - c_t.$$

Note that TFP A_t affects both the production possibilities frontier of the economy (the resource constrain) and the incentives to accumulate capital (the Euler condition).

- In the steady state, both k^* and c^* are increasing in A.

A. Unanticipated Permanent Productivity Shock

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, permanently.

- c_0 may either increase or fall, depending on whether wealth or substitution effect dominates. Along the transition, both c and k are increasing towards the new higher steady state. See **Figure 5** for the dynamics.

B. Unanticipated Transitory Productivity Shock

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, but only for $t \in [0, \widehat{t}]$ for some finite $\widehat{t}$.

- Again, c_0 may either increase or fall, depending on whether wealth or substitution effects dominates. I consider the case that c_0 increases. A typical transition is depicted in **Figure 6**.

3.5.4 Government Spending (Figure 7 and 8)

- We now introduce a government that collects taxes in order to finance some exogenous level of government spending.

A. Lump Sum Taxation

- Suppose the government finances its expenditure with lump-sum taxes. The household budget is

$$c_t^j + k_{t+1}^j = w_t + r_t k_t^j + (1-\delta)k_t^j - T_t,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + r_{t+1} - \delta] = \beta[1 + f'(k_{t+1}) - \delta]$$

That is, taxes do not affect the savings choice. On the other hand, the government budget is

$$T_t = g_t,$$

where g_t denotes government spending. The resource constraint of the economy becomes

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t - g_t$$

- In the steady state, k^* is independent of g and c^* moves one-to-one with $-g$. Along the transition, a permanent increase in g both decreases c and slows down capital accumulation. See **Figure 7.**

- Note that the effect of government spending financed with lump-sum taxes is isomorphic to a negative endowment shock.

B. Distortionary Taxation

- Suppose the government finances its expenditure with distortionary income taxation. The household budget is

$$c_t^j + k_{t+1}^j = (1-\tau)(w_t + r_t k_t^j) + (1-\delta)k_t^j,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + (1-\tau)r_{t+1} - \delta] = \beta[1 + (1-\tau)f'(k_{t+1}) - \delta].$$

That is, taxes now distort the savings choice. On the other hand, the government budget is

$$g_t = \tau f(k_t)$$

and the resource constraint of the economy is again

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = (1-\tau)f(k_t) - \delta k_t - c_t.$$

Government spending is now isomorphic to a negative TFP change.

- In the steady state, k^* is a decreasing function of g (equivalently, τ) and c^* decreases more than one-to-one with g. Along the transition, a permanent increase in g (and τ) drastically slows down capital accumulation. The immediate See **Figure 7.**

- Note that the effect of government spending financed with distortionary taxes is isomorphic to a negative TFP shock.

3.6 Endogenous Labor Supply, the RBC Propagation Mechanism, and Beyond

3.6.1 The Phase Diagram with Endogenous Labor Supply

- Solve for labor supply as a function of k and c :

$$\frac{U_x(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t) \Rightarrow l_t = l(k_t, c_t)$$

  Note that l increases with k, but less than one-to-one (or otherwise F_L would fall). This reflects the substitution effect. On the other hand, l falls with c, reflecting the wealth effect.

- Substitute back into the dynamic system for k and c, assuming CEIS preferences:

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t/l(k_t, c_t)) - \delta - \rho],$$

$$\dot{k}_t = f(k_t, l(k_t, c_t)) - \delta k_t - c_t,$$

  which gives a system in k_t and c_t alone.

- Draw suggestive phase diagram. See Figure **??**.

- Note that the $\dot{c}$ is now negatively sloped, not vertical as in the model with exogenously fixed labor. This reflects the wealth effect on labor supply. Lower c corresponds to lower effective wealth, which results to higher labor supply for any given k (that is, for any given wage).

3.6.2 Impulse Responces Revisited

- Note that the endogeneity of labor supply makes the Euler condition (the $\dot{c}$ locus) sensitive to wealth effects, but also mitigates the impact of wealth effects on the resource constraint (the $\dot{k}$ locus).

- Reconsider the impulse responces of the economy to shocks in productivity or goverment spending.

- Government spending.... If financed with lump sum taxes, an increase in g has a negative wealth effect, which increases labor supply. This in turn leads an increase in the MPK and stimulates more investment. At the new steady state the capital-labor ratio remains the same, as it is simply the one that equates the MPK with the discount rate, but both employment and the stock of capital go up...

- Note that the above is the supply-side effect of government spending. Contrast this with the demand-side effect in Keynesian models (e.g., IS-LM).

- Productivity shocks....

3.6.3 The RBC Propagation Mechanism, and Beyond

- Just as we can use the model to "explain" the variation of income and productivity levels in the cross-section of countries (i.e., do the Mankiw-Romer-Weil exercise), we can also use the model to "explain" the variation of income, productivity, investment and employment in the time-series of any given country. Hence, the RBC paradigm.

- The heart of the RBC propagation mechanism is the interaction of consumption smoothing and deminishing returns to capital accumulation. Explain....

- This mechanism generates endogenous *persistence* and *amplification*. Explain...

- Enogenous persistence is indeed the other face of conditional convergence. But just as the model fails to generate a substantially low rate of conditional convergence, it also fails to generate either substantial persistence or substantial amplification. For the model to match the data, we then need to assume that exogenous productivity (the Solow residual) is itself very volatile and persistent. But then we partly answer and partly peg the question.

- Hence the search for other endogenous propagation mechanisms.

- Discuss Keynesian models and monopolistic competition... Discuss the potential role financial markets...

to be completed

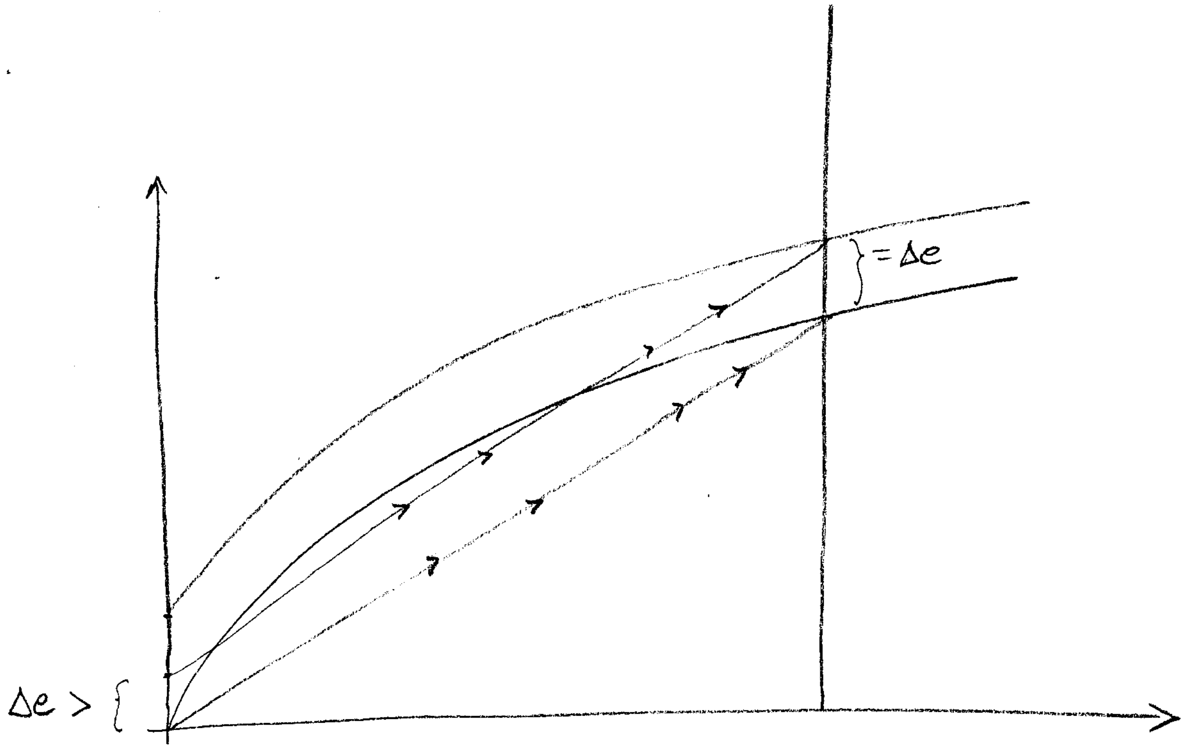


FIGURE 2

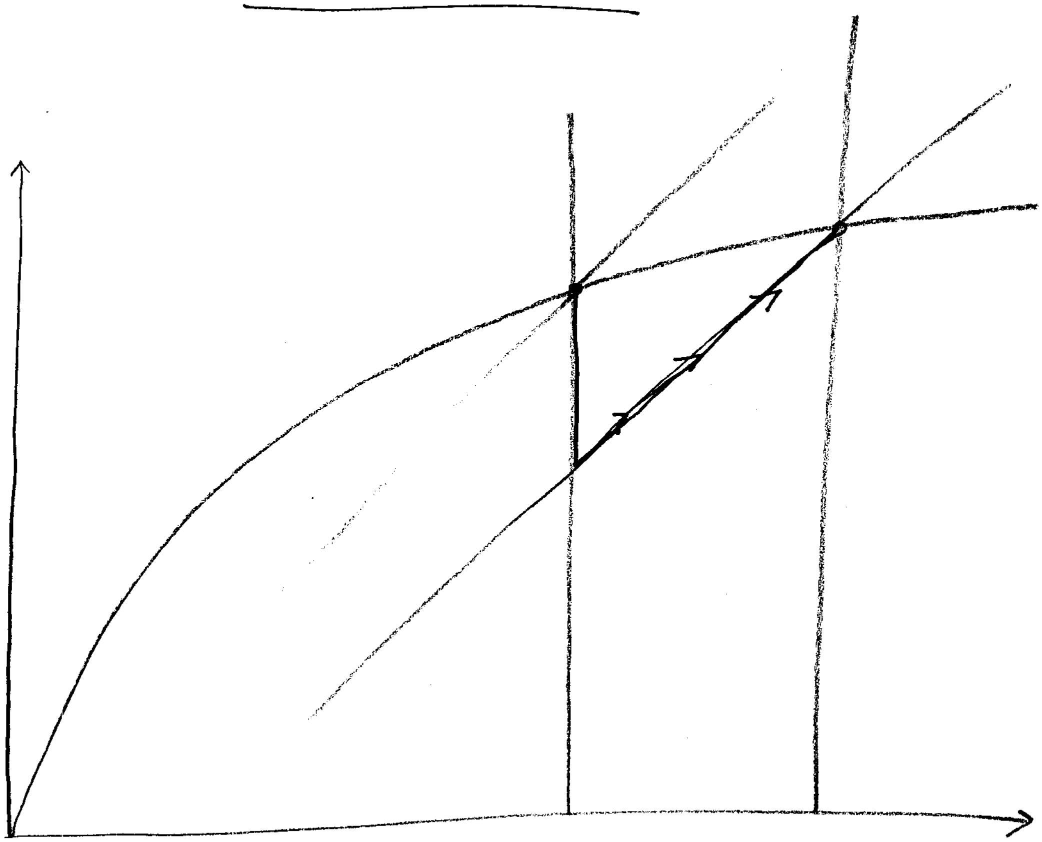


FIGURE 3

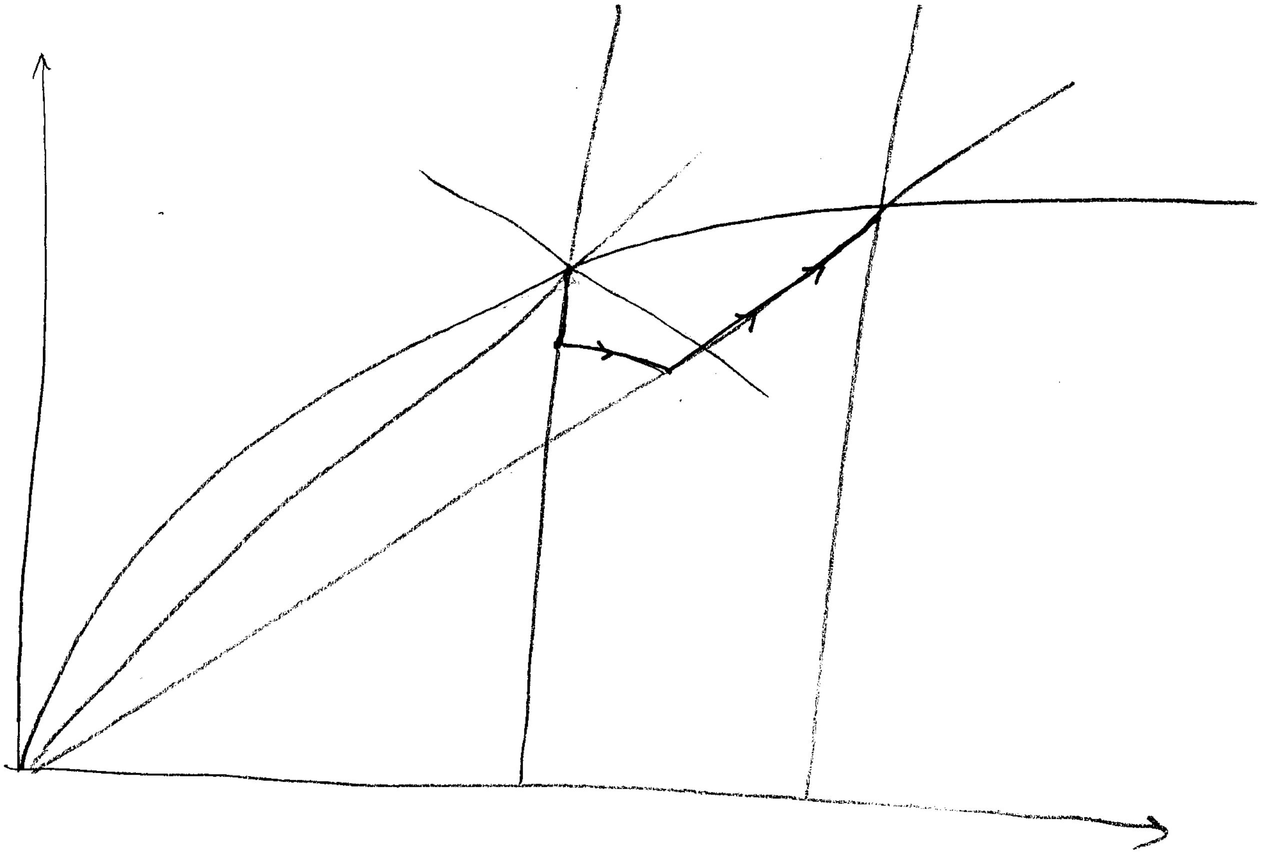


FIGURE 4

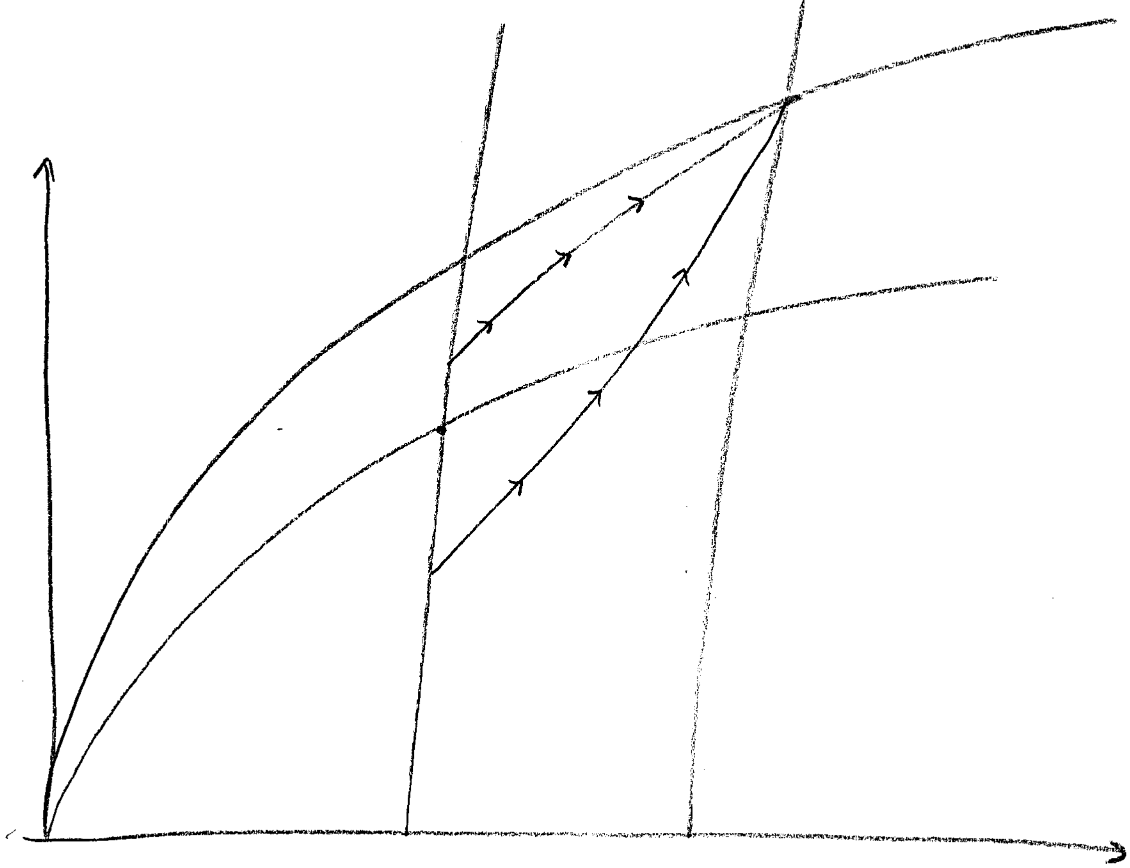
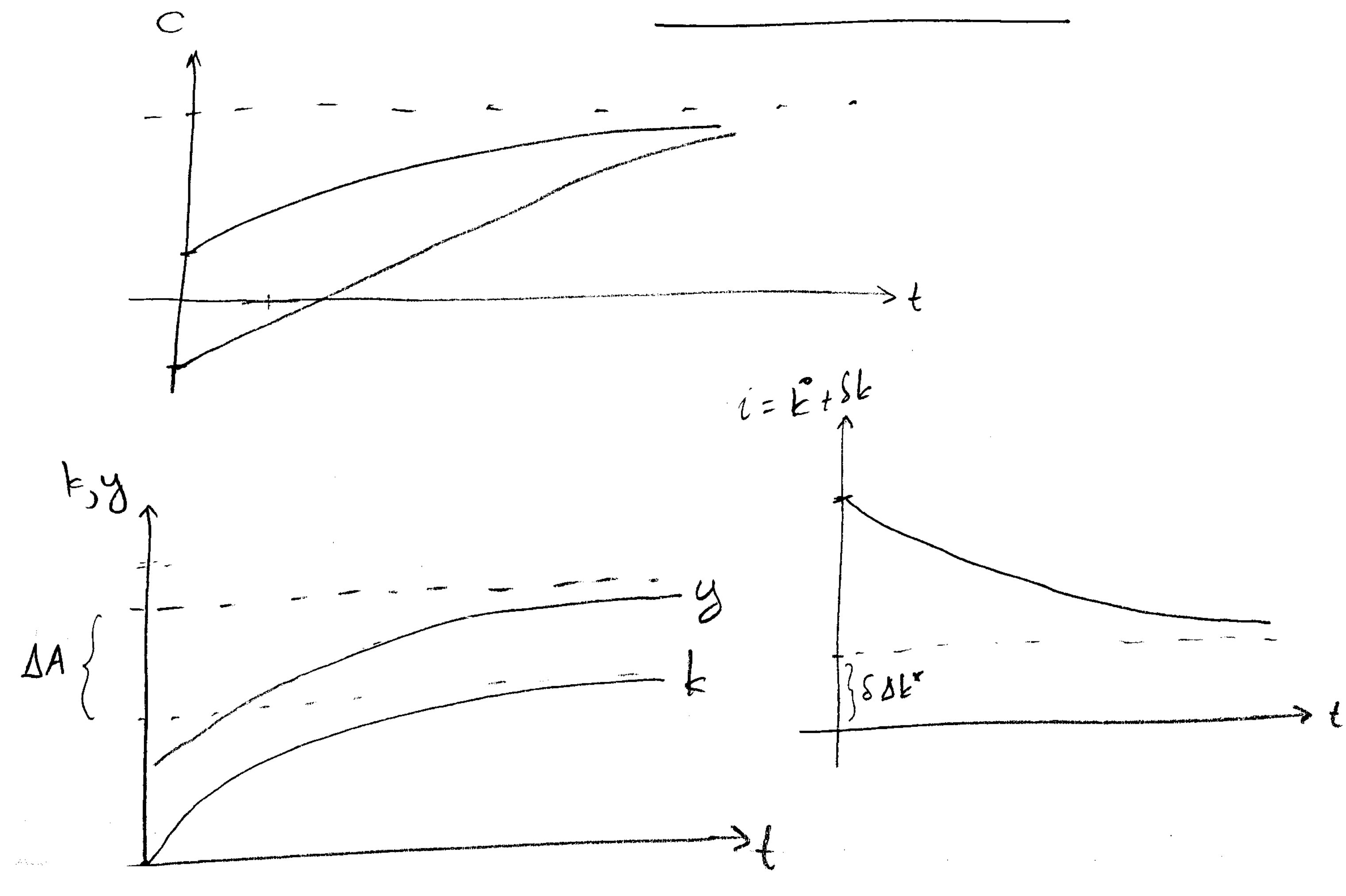


FIGURE 5



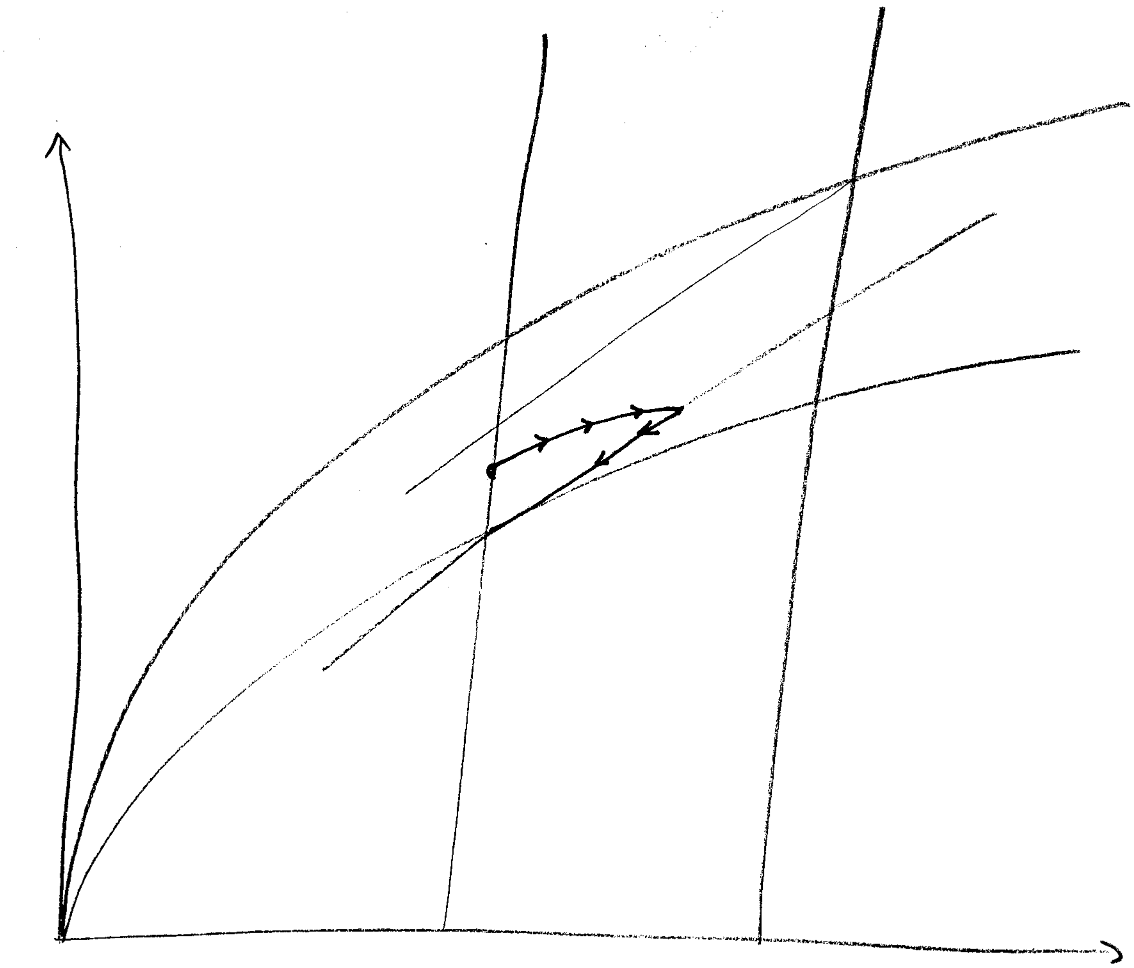


FIGURE 6