14.102 Midterm Exam

October 20, 2005

Instructions: This is a closed book exam; you have 90 minutes to complete it. Please answer all questions.

Suggestion: Read through all questions briefly before you being, and pay attention to the allocation of points.

Good luck!

Real Analysis (15 points)

1. (2 points) Give an example of a bounded sequence with exactly three limit points.
2. (2 points) True or false: every sequence with a limit point is bounded.
3. (6 points) Let $\{A\}$ be a (possibly infinite) collection of convex sets, where $A' \subseteq \mathbb{R}^n$ for all $A' \in \{A\}$. Show that the intersection of all the sets in $\{A\}$ is itself a convex set.
4. (5 points) Let $f : U \to \mathbb{R}$ be defined on a convex subset $U \subseteq \mathbb{R}$, and let f be quasiconcave. Define the relation R by

$$\forall x, y \in U, \; xRy \Longleftrightarrow f(x) \geq f(y)$$

Show that $\forall x, y, z \in U$ and $\lambda \in [0, 1]$, if xRz and yRz, then $(\lambda x + (1 - \lambda)y)\, Rz$.

Linear Algebra (45 points)

5. Let B be the matrix $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
   (a) (3 points) Find $\det(B)$
   (b) (2 points) Find $rank(B)$
   (c) (2 points) Find B^{-1}, if it exists.
   (d) (6 points) Is B diagonalizable? If so, diagonalize B; that is, find diagonal matrix Λ and nonsingular matrix V such that $B = V \Lambda V^{-1}$.
   (e) (3 points) Can B be diagonalized such that $B = V \Lambda V'$, i.e. such that $V' = V^{-1}$? You do not need to find such V.
   (f) (2 points) Show that B is idempotent.
   (g) (3 points) Is B positive semidefinite, negative semidefinite, or indefinite? (Hint: no calculations necessary.)

6. Now let P be any symmetric and idempotent $n \times n$ matrix, and let I be the $n \times n$ identity matrix.

   (a) (4 points) Show that the matrix $I - P$ is symmetric.

   (b) (4 points) Show that $I - P$ is idempotent.

   (c) (4 points) Show that P and $I - P$ are orthogonal.

7. Consider the following system of equations, where $a_{33}, b_1, b_2, b_3, x_1, x_2$, and x_3 are real numbers:

$$\begin{aligned} 3x_1 + 6x_2 &= b_1 \\ 6x_1 + 3x_2 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

   (a) (3 points) Rewrite the system in matrix notation, as $Ax = b$.

   (b) For each of the following statements, provide conditions on A and/or b such that the statement is true:

      i. (3 points) The system has a unique solution.

      ii. (3 points) The system has no solution.

      iii. (3 points) The system has multiple solutions.

Optimization in $\mathbb{R}^n$ (40 points)

8. Consider the standard consumer utility maximization problem over two goods, subject to a linear budget constraint and nonnegative consumption:

$$\begin{aligned} \max u(x,y) & \\ s.t.\ p_x x + p_y y &\leq I \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

   (a) (3 points) Assume throughout that $u : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $I > 0$. What restrictions on the parameters p_x and p_y are sufficient for the Weierstraß theorem to guarantee a solution to this maximization problem?

   (b) (5 points) Assume that these restrictions hold. Show that the rank constraint qualification holds at any feasible point.

   (c) (4 points) Consider now the dual expenditure minimization problem:

$$\begin{aligned} \min p_x x + p_y y & \\ s.t.\ u(x,y) &\geq \overline{U} \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

   Under what conditions does the level set $u(x,y) = \overline{U}$ *not* define y as a function of x?

(d) Assume that the problem is such that an *interior* solution (x^*, y^*) exists and $u(x, y) = \overline{U}$ (the constraint binds) .

   i. (4 points) What is $y'(x^*)$, in terms of the utility function (where $y(x)$ is y implicitly defined as a function of x along the level set $u(x, y) = \overline{U}$)?

   ii. (4 points) What is $y'(x^*)$, in terms of the prices p_x and p_y?

   iii. (4 points) *Briefly* discuss these results in terms of the theorem of Lagrange and economic intuition.

9. Consider the following constrained optimization problem:

$$\begin{aligned} \max 3xy - x^3 & \\ s.t.\ 2x - y &= -5 \\ 5x + 2y &\geq 37 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(a) (8 points) Write out the Lagrangian and derive its first order conditions, as well as the complementary-slackness conditions.

(b) (8 points) Find the solution to the maximization problem.