14.102 Midterm Exam Solutions

October 20, 2005

Real Analysis (15 points)

1. (2 points) Give an example of a bounded sequence with exactly three limit points.

   Solution: One example is the sequence $\{1, 2, 3, 1, 2, 3, 1, 2, 3, ...\}$, which has the integers 1,2, and 3 as limit points.

2. (2 points) True or false: every sequence with a limit point is bounded.

   Solution: False. For example, the sequence $\{1, 1, 1, 2, 1, 3, 1, 4, ...\}$ - i.e., $a_n = 1$ if n is odd and $a_n = \frac{n}{2}$ if n is even - has 1 as a limit point, but is not bounded.

3. (6 points) Let $\{A\}$ be a (possibly infinite) collection of convex sets, where $A' \subseteq \mathbb{R}^n$ for all $A' \in \{A\}$. Show that the intersection of all the sets in $\{A\}$ is itself a convex set.

   Solution: Let $B = \cap\{A\}$, the intersection of the sets in $\{A\}$. Then B is convex if for any $x, y \in B$ and $\alpha \in \mathbb{R}$, $\alpha x + (1-\alpha)y \in B$. So take arbitrary $x, y \in B$. Because B is the intersection of all the sets in $\{A\}$, x and y are both elements of every set in $\{A\}$. For example, if $A'' \in \{A\}$, then $x, y \in A''$. Because A'' is convex, $\alpha x + (1-\alpha)y \in A''$. But because A'' was an arbitary element of $\{A\}$, this shows that $\alpha x + (1-\alpha)y \in A'$ for all $A' \in \{A\}$. This implies that $\alpha x + (1-\alpha)y \in B$, as was to be shown.

4. (5 points) Let $f : U \to \mathbb{R}$ be defined on a convex subset $U \subseteq \mathbb{R}$, and let f be quasiconcave. Define the relation R by

   $$\forall x, y \in U,\ xRy \Longleftrightarrow f(x) \geq f(y)$$

   Show that $\forall x, y, z \in U$ and $\lambda \in [0, 1]$, if xRz and yRz, then $(\lambda x + (1-\lambda)y)\, Rz$.

   Solution: Without loss of generality, suppose xRy, i.e. that $f(x) \geq f(y)$. Then f quasiconcave and the hypothesis imply that $f(\lambda x + (1-\lambda)y) \geq f(y) \geq f(z)$, i.e. that $(\lambda x + (1-\lambda)y)\, Rz$.

Linear Algebra (45 points)

5. Let B be the matrix $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

   (a) (3 points) Find $\det(B)$

   Solution: $\det(B) = \frac{1}{4} - \frac{1}{4} = 0$

   (b) (2 points) Find $rank(B)$

   Solution: $rank(B) = 1$; this is clear either from the fact that its second column is simply the opposite of its first, or from the fact that its determinant is zero, and that it is therefore of less than full rank.

(c) (2 points) Find B^{-1}, if it exists.

Solution: It does not exist, because B is not full rank.

(d) (6 points) Is B diagonalizable? If so, diagonalize B; that is, find diagonal matrix Λ and nonsingular matrix V such that $B = V\Lambda V^{-1}$.

Solution: To diagonalize B, we first find its eigenvalues. We already know that one of them is zero (because B itself is singular). To find the other, we write $\begin{vmatrix} \frac{1}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} = \frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} = \lambda^2 - \lambda = 0 \Longrightarrow \lambda = 0$ or 1; one is the other eigenvalue. Therefore, the diagonal matrix Λ is simply $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. To find V we need to find eigenvectors associated with these eigenvalues. For eigenvalue 1, we need a solution to $\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Clearly, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will solve this equation. For eigenvalue 0, we need a solution to .$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and one solution is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So our matrix V can be $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (note that the order of the eigenvectors is determined by the placement of the eigenvalues in Λ). $V^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, so we can diagonalize B as $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(e) (3 points) Can B be diagonalized such that $B = V\Lambda V'$, i.e. such that $V' = V^{-1}$? You do not need to find such V.

Solution: Yes; because B is symmetric we can find an orthonormal diagonalization.

(f) (2 points) Show that B is idempotent.

Solution: This is simply a matter of checking $BB = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = B$.

(g) (3 points) Is B positive semidefinite, negative semidefinite, or indefinite? (Hint: no calculations necessary.)

Solution: Any matrix which is symmetric and idempotent is positive semidefinite. The proof (which was not required for this question) is simple: for symmetric and idempotent A, $x'Ax = x'AAx = x'A'Ax = y'y = \sum_{i=1}^{n} y_i^2 \geq 0$, where $y = Ax$.

6. Now let P be any symmetric and idempotent $n \times n$ matrix, and let I be the $n \times n$ identity matrix.

Note: if you haven't already, you'll soon see these sorts of matrices a lot in econometrics. P will be a symmetric and idempotent projection matrix (i.e.,

$X(X'X)^{-1}X'$, which you saw in problem set 2 and which is the projection matrix onto the space spanned by X) and $I - P$ will be what Greene refers to as the 'residual maker', and is also a projection matrix onto the null space of P.

(a) (4 points) Show that the matrix $I - P$ is symmetric.
Solution: $(I - P)' = I' - P' = I - P$

(b) (4 points) Show that $I - P$ is idempotent.
Solution: $(I - P)(I - P) = I - P - P + PP = I - P - P + P = I - P$

(c) (4 points) Show that P and $I - P$ are orthogonal.
Solution: $P'(I - P) = P' - P'P = P - PP = P - P = 0$; similarly, $(I - P)'P = P - P'P = P - PP = P - P = 0$.
Note: Most of you showed that $P(I - P) = 0$. In general, to show that A and B are orthogonal, you want to show that $A'B = 0$, i.e. that the columns of A and B are orthogonal to each other. Note that $AB = 0$ means that the rows of A are orthogonal to the rows of B, and might not even be defined; A and B orthogonal should be a well-defined concept for A $m \times n$ and B $m \times k$ (so that their columns are elements of the same vector space). Here, because P is symmetric, $P = P'$ and so $P'(I - P) = 0 \Leftrightarrow P(I - P) = 0$.

7. Consider the following system of equations, where $a_{33}, b_1, b_2, b_3, x_1, x_2$, and x_3 are real numbers:

$$\begin{aligned} 3x_1 + 6x_2 &= b_1 \\ 6x_1 + 3x_2 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

(a) (3 points) Rewrite the system in matrix notation, as $Ax = b$ (that is, what are A, x and b?).
Solution: $\begin{bmatrix} 3 & 6 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b) For each of the following statements, provide conditions on A and/or b such that the statement is true:

i. (3 points) The system has a unique solution.
Solution: If $a_{33} \neq 0$, then $rank[A, b] = rank[A] = 3$; in this case, A is invertible and the system has the unique solution $x = A^{-1}b$.

ii. (3 points) The system has no solution.
Solution: If $a_{33} = 0$ and $b_3 \neq 0$, then $rank[A, b] > rank[A] = 2$, then the system has no solution.

iii. (3 points) The system has multiple solutions.
Solution: If $a_{33} = 0$ and $b_3 = 0$, then $rank[A, b] = rank[A] = 2 < 3$, and the system has multiple solutions.

Optimization in $\mathbb{R}^n$ (40 points)

8. Consider the standard consumer utility maximization problem over two goods, subject to a linear budget constraint and nonnegative consumption:

$$\begin{aligned} \max u(x,y) & & \\ s.t.\ p_x x + p_y y &\leq I \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

   (a) (3 points) Assume throughout that $u : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $I > 0$. What restrictions on the parameters p_x and p_y are sufficient for the Weierstraß theorem to guarantee a solution to this maximization problem?

   Solution: It is sufficient that they be strictly positive, such that the region described by the three constraints is a compact set in the positive quadrant.

   (b) (5 points) Assume that these restrictions hold. Show that the rank constraint qualification holds at any feasible point.

   Solution: The rank constraint qualification holds if $[\nabla h^*(x^*, y^*)]$, the matrix whose colums are the gradients of the constraints which *bind* at an optimum, has full rank at that optimum. The matrix of the gradients of the constraints is $\begin{bmatrix} p_x & 1 & 0 \\ p_y & 0 & 1 \end{bmatrix}$. Any two columns of this matrix are linearly independent. Because no more than two of the constraints can bind at an optimum (if $x = y = 0$ then $p_x x + p_y y < I$), the constraint qualification will hold.

   (c) (4 points) Consider now the dual expenditure minimization problem:

$$\begin{aligned} \min p_x x + p_y y & & \\ s.t.\ u(x,y) &\geq \overline{U} \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

   Under what conditions does the level set $u(x, y) = \overline{U}$ *not* define y as a function of x?

   Solution: This is the case at any point (x_0, y_0) such that $\frac{\partial u}{\partial y}(x_0, y_0) = 0$. Note that many of the functional forms we employ for utility functions rule this out.

   (d) Assume that the problem is such that an *interior* solution (x^*, y^*) exists and $u(x, y) = \overline{U}$ (the constraint binds) .

      i. (4 points) What is $y'(x^*)$, in terms of the utility function (where $y(x)$ is y implicitly defined as a function of x along the level set $u(x, y) = \overline{U}$)?

      Solution: $y'(x^*) = -\frac{\partial u/\partial x}{\partial u/\partial y}(x^*, y^*)$, according to the implicit function theorem.

ii. (4 points) What is $y'(x^*)$, in terms of the prices p_x and p_y?
Solution: $y'(x^*) = -\frac{p_x}{p_y}$

iii. (4 points) *Briefly* discuss these results in terms of the theorem of Lagrange and economic intuition.
Solution: The theorem of Lagrange tells us that at an interior local optimum, the gradient of the objective function and that of the constraint set are scalar multiples of one another, which in turn implies that the slopes of their level sets must be equal, as we see in parts (i) and (ii). The economic intuition of the problem is that at an interior optimum, the consumer should consume a bundle such that the rate at which he is willing to trade one good for the other is just equal to the ratio of their prices.

9. Consider the following constrained optimization problem:

$$\begin{aligned} \max 3xy - x^3 & \\ s.t.\ 2x - y &= -5 \\ 5x + 2y &\geq 37 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(a) (8 points) Write out the Lagrangian and derive its first order conditions, as well as the complementary-slackness conditions.
Solution: The Lagrangian is

$$L(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 3xy - x^3 - \lambda_1(2x - y + 5) + \lambda_2(5x + 2y - 37) + \lambda_3 x + \lambda_4 y$$

The first order and complementary slackness conditions are

$$\begin{aligned} \frac{\partial L}{\partial x} &= 3y - 3x^2 - 2\lambda_1 + 5\lambda_2 + \lambda_3 = 0 \\ \frac{\partial L}{\partial y} &= 3x + \lambda_1 + 2\lambda_2 + \lambda_4 = 0 \\ \frac{\partial L}{\partial \lambda_1} &= -2x + y - 5 = 0 \\ \lambda_2(5x + 2y - 37) &= \lambda_3 x = \lambda_4 y = 0 \\ \lambda_2 &\geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0 \end{aligned}$$

Note: another way to solve this problem was to begin by substituting $y = 2x + 5$ throughout; solutions of this sort are fine.

(b) (8 points) Find the solution to the maximization problem.
Solution: If $x = 0$, the function is equal to zero, and if $y = 0$ the function is nonpositive; we will begin by assuming that neither of the constraints $x \geq 0$ or $y \geq 0$ bind (note that if $x > 0$, $y = 2x + 5 > 0$). This implies that

$\lambda_3 = \lambda_4 = 0$. We will now proceed by examining separately the cases in which which $5x + 2y \geq 37$ does and does not bind. First, assume that it does not. This implies that $\lambda_2 = 0$, and reduces our system to the following three equations in three unknowns:

$$\begin{aligned} 3y - 3x^2 - 2\lambda_1 &= 0 \qquad (1) \\ 3x + \lambda_1 &= 0 \qquad (2) \\ 2x + 5 &= y \qquad (3) \end{aligned}$$

Plugging (2) and (3) into (1) we have

$$\begin{aligned} 6x + 15 - 3x^2 + 6x &= 0 \qquad (4) \\ 3x^2 - 12x - 15 &= 0 \qquad (5) \\ x^2 - 4x - 5 &= 0 \qquad (6) \\ (x-5)(x+1) &= 0 \qquad (7) \end{aligned}$$

which has roots $x = 5, x = -1$. We can rule out $x = -1$ because it doesn't satisfy the constraint $x \geq 0$. If $x = 5, y = 2x + 5 = 15$; our first critical point is $(5, 15)$; $f(5, 15) = 100$. Note that we can now say (because we have found a point where $f(x, y) > 0$) that our intitial assumptions that $x > 0$ and $y > 0$ were correct.

Note also that at this optimum, $5x + 2y = 25 + 30 = 55 > 37$, and so the assumption that $\lambda_2 = 0$ was justified as well. But we can also check that $\lambda_2 > 0$: then $5x + 2y = 37$. In this case, a candidate solution must satisfy

$$\begin{aligned} 2x + 5 &= y \qquad (8) \\ 5x + 2y &= 37 \qquad (9) \end{aligned}$$

which has the unique solution $x = 3, y = 11$; $f(3, 11) = 72 < 100$.

We have shown that $(5, 15)$ is the maximizer among the critical points of the Lagrangian. We must still verify that the point $(5, 15)$ is indeed the global maximum. The Hessian is $\begin{bmatrix} -6x & 3 \\ 3 & 0 \end{bmatrix}$; the leading principal minors are $-6x$ and -9, both of which are strictly negative when $x = 5$, implying that $(5, 15)$ is a local maximum. Moreover, note that because $y = 2x + 5$, $f = 6x^2 + 15x - x^3$. This function goes to negative infinity as x (and y) go to infinity; we thus have a global maximum at $(5, 15)$.