Handout on the eigenvectors of distinct eigenvalues
9/30/04

This handout shows, first, that eigenvectors associated with distinct eigenvalues of an abitrary square matrix are linearly indpenent, and second, that all eigenvectors of a symmetric matrix are mutually orthogonal.

First we show that all eigenvectors associated with distinct eigenvalues of an abitrary square matrix are mutually linearly independent:

Suppose k $(k \leq n)$ **eigenvalues** $\{\lambda_1, ..., \lambda_k\}$ **of** A **are distinct, and take any corresponding eigenvectors** $\{v_1, ..., v_k\}$, **defined by** $v_j \neq 0, Av_j = \lambda_j v_j$ **for** $j = 1, ..., k$. **Then,** $\{v_1, ..., v_k\}$ **are linearly independent.**

First consider two such eigenvectors. Suppose we have eigenvalue λ with eigenvector v, and eigenvalue μ with eigenvector w, $\lambda \neq \mu$. We will show that $\alpha v + \beta w = 0 \Rightarrow \alpha = \beta = 0$, implying that v and w are linearly independent.

So suppose we have

$$\alpha v + \beta w = 0 \tag{1}$$
$$\alpha A v + \beta A w = 0 \tag{2}$$
$$\alpha \lambda v + \beta \mu w = 0 \tag{3}$$

Now multiply (1) by $-\lambda$ and add to (3) to get

$$\beta(\mu - \lambda) w = 0 \tag{4}$$

This implies that $\beta = 0$, and plugging this back into (1) implies that $\alpha = 0$ as well.

Now consider any three eigenvectors (v, w and u) with with distinct eigenvalues (λ, μ, and ν); we proceed in much the same manner:

$$\alpha v + \beta w + \gamma u = 0 \tag{5}$$
$$\alpha A v + \beta A w + \gamma A u = 0 \tag{6}$$
$$\alpha \lambda v + \beta \mu w + \gamma \nu u = 0 \tag{7}$$
$$\beta(\mu - \lambda) w + \gamma(\nu - \lambda) u = 0 \tag{8}$$

But this is a linear combination of two eigenvectors, and we have just shown that they must be linearly independent. So we have $\beta = \gamma = 0$, which implies that $\alpha = 0$ as well.

We can continue in this manner to show that any k eigenvectors with distinct eigenvalues are linearly indpendent.

Now we want to show that all the eigenvectors of a symmetric matrix are mutually orthogonal.

Suppose k $(k \leq n)$ **eigenvalues** $\{\lambda_1, ..., \lambda_k\}$ **of** A **are distinct with** A **symmetric, and take any corresponding eigenvectors** $\{v_1, ..., v_k\}$, **defined by** $v_j \neq 0, Av_j = \lambda_j v_j$ **for** $j = 1, ..., k$. **Then,** $\{v_1, ..., v_k\}$ **are orthogonal.**

To see this, suppose v is an eigenvector for λ and w is an eigenvector for μ. Then

$$w'Av = \lambda w'v$$

and

$$w'Av = \mu w'v$$

(where the second equation is derived by taking the transpose of $Aw = \mu w$ and postmultiplying by v). Thus,

$$0 = (\lambda - \mu)w'v$$

implying that $w'v = 0$, or that w and v are orthogonal.

If we have repeated eigenvalues, we can still find mutually orthogonal eigenvectors (though not every set of eigenvectors need be orthogonal).

We have

$$\begin{aligned} Av &= \lambda v \\ Aw &= \lambda w \end{aligned}$$

It is not necessarily true that $w'v = 0$ for arbitrary solutions to these equations; however, we can choose a linear combination of v and w which is still an eigenvector, and which is orthogonal to w. Simply multiply the first equation above by α, and the second by β (two scalars), and add to get

$$A(\alpha v + \beta w) = \lambda(\alpha v + \beta w)$$

Clearly $(\alpha v + \beta w)$ is an eigenvector. Now we want to choose α and β so that $(\alpha v + \beta w)'w = 0$:

$$\begin{aligned} (\alpha v' + \beta w')w &= 0 \\ \alpha v'w + \beta w'w &= 0 \\ \alpha &= -\frac{\beta w'w}{v'w} \end{aligned}$$

Which exists as long as $v'w \neq 0$, which is the case we were concerned with.