

14.102, Math for Economists
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These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples, mostly from Rudin. The usual disclaimer applies; questions and comments are welcome.

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1 Real Analysis I - Basic Set Theory

We begin from the fundamental notion of a set, which is simply a collection of... well, anything (but for us it's usually numbers, functions, spaces, metrics, etc.).

We use standard notation $A \cap B$ for the intersection of two (or more) sets, $A \cup B$ for union, A^c for complement (i.e., for the set of all elements not in A). More precisely, if A and B are subsets of a set X (denoted $A \subset X$ and $B \subset X$) with typical element x :

$$\begin{aligned}A \cap B &= \{x : x \in A \text{ and } x \in B\} \\A \cup B &= \{x : x \in A \text{ or } x \in B\} \\A^c &= \{x : x \in X, x \notin A\}\end{aligned}$$

If $A \subset B$ and $B \subset A$, we say $A = B$ (indeed, it is often easiest to prove that two sets are equal by proving these two steps).

1.1 Relations and Equivalences

For two sets A and B , a *relation* between two points is a function $R : A \times B \rightarrow \{0, 1\}$. We write xRy if points x and y are in relation R (i.e., $R(x, y) = 1$).

Example 1 xRy if $x_1 > y_1$

Example 2 xRy if $x_1 = y_2$

Example 3 xRy if $\|x\| = \|y\|$

As it turns out, there is one fundamental class of relations that is important in microeconomic theory. These are called *equivalence relations* or *equivalences*.

Definition 4 A relation is called *equivalence* (usually denoted by \sim) if it satisfies the following three properties:

- $x \sim x$ (*reflexive*)
- $x \sim y \implies y \sim x$ (*symmetric*)
- $x \sim y \ \& \ y \sim z \implies x \sim z$ (*transitive*)

Exercise 5 For each of the three examples of relations above, find out whether it is reflexive, symmetric and transitive.

Exercise 6 Give an example of R that is symmetric but not transitive.

Equivalence relations are essential for an axiomatic development of the utility function: for a utility function to exist, it is a necessary condition that relation “ xRy if the consumer is indifferent between bundles x and y ” be an equivalence (why?). Although it seems obvious that this relation is indeed an equivalence, and economic models usually assume that it is, it might not be a great description of reality. For instance, I am pretty much indifferent between my welfare now and if I give away a nickel; however, such indifference is surely not transitive: if I give away a million nickels, I will be significantly worse off.

Definition 7 Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

1. If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true.
2. If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Note that the relation $<$ is transitive, but neither reflexive nor symmetric (indeed, it is *antisymmetric*: $x < y \implies y \not< x$).

It is often convenient to write $x > y$ in place of $y < x$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of the two holds. In other words, $x \leq y$ is the negation of $x > y$.

1.2 Ordered Sets

Definition 8 An **ordered set** is a set S for which an order is defined.

Definition 9 Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E .

Lower bounds are defined in the same way, with \geq in place of \leq .

Definition 10 Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

1. α is an upper bound of E .
2. If $\gamma < \alpha$ then γ is not an upper bound of E .

Then α is called the **least upper bound of E** or the **supremum of E** , and we write $\alpha = \sup E$.

The **greatest lower bound**, or **infimum**, of a set E which is bounded below is defined in the same manner: The statement $\alpha = \inf E$ means that α is a lower bound of E and that no $\beta > \alpha$ is a lower bound of E .

Definition 11 An ordered set S is said to have the **least-upper-bound property** if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

Exercise 12 Show that \mathbb{Q} does not have the least-upper-bound property.

Theorem 13 Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then $\alpha = \sup L$ exists in S , and $\alpha = \inf B$. In particular, $\inf B$ exists in S .

Proof. Since B is bounded below, L is not empty. By the definition of L , we see that every $x \in B$ is an upper bound of L . Thus L is bounded above. Thus, $\sup L$ exists in S ; call it α .

If $\gamma < \alpha$, then γ is not an upper bound of L , so $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

Finally, note that any $\beta > \alpha$ is not in L , because α is an upper bound for L .

Thus, we have shown that $\alpha \in L$, but that any $\beta > \alpha$ is not in L . But L is the set of all lower bounds for B , so α is the greatest lower bound for B . This means precisely that $\alpha = \inf B$. ■

1.3 Finite, Countable, and Uncountable Sets

Definition 14 For any positive integer n , let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A , we say:

1. A is **finite** if $A \sim J_n$ for some n (the empty set is also considered to be finite).
2. A is **infinite** if A is not finite.
3. A is **countable** if $A \sim J$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if A is finite or countable.

Note that by convention, countable implies infinite (so, strictly speaking, we do not say 'countably infinite', although you will hear this phrase from time to time).

For two finite sets, we have $A \sim B$ iff A and B 'have the same number of elements'. But for infinite sets this notion becomes vague, while the idea of 1-1 correspondence (under which, given a mapping from A to B , the image in B of $x_1 \in A$ is distinct from the image in B of $x_2 \in A$ whenever x_1 is distinct from x_2) retains its clarity.

Example 15 *Let A be the set of all integers. Then A is countable. For consider the following arrangement of the sets A and J :*

$$A : 0, 1, -1, 2, -2, 3, -3, \dots$$

$$J : 1, 2, 3, 4, 5, 6, 7, \dots$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}) \\ -\frac{n-1}{2} & (n \text{ odd}) \end{cases}$$

Theorem 16 *Every infinite subset of a countable set A is countable.*

Proof. *Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:*

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen n_1, \dots, n_{k-1} ($k = 2, 3, 4, \dots$), let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Then, letting $f(k) = x_{n_k}$ ($k = 1, 2, 3, \dots$), we obtain a 1-1 correspondence between E and J . ■

One interpretation of the theorem is that countability represents the 'smallest' kind of infinity, in that no uncountable set can be a subset of a countable set.

Theorem 17 *Let $\{E_n\}$ be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Proof. *Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \dots$ and consider the infinite array*

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

in which the elements of E_n form the n^{th} row. The array contains all elements of S . We can arrange these elements in a sequence as follows:

$$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}, \dots$$

If any of the sets E_n have elements in common, these will appear more than once in the above sequence. Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable. Since $E_1 \subset S$, and E_1 is infinite, S is also infinite, and thus countable. ■

Theorem 18 Let A be a countable set, and let B_n be the set of all n -tuples (a_1, \dots, a_n) where $a_k \in A$ ($k = 1, 2, \dots, n$) and the elements a_1, \dots, a_n need not be distinct. Then B_n is countable.

Proof. That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable ($n = 2, 3, 4, \dots$). The elements of B_n are of the form

$$(b, a) \quad (b \in B_{n-1}, a \in A)$$

For ever fixed b , the set of pairs (b, a) is equivalent to A , and thus countable. Thus B_n is the union of a countable set of countable sets; thus, B_n is countable, and the proof follows by induction on n . ■

Corollary 19 The set of all rational numbers is countable.

Proof. We apply the previous theorem with $n = 2$, noting that every rational number can be written as b/a , where b and a are integers. Since the set of pairs (b, a) is countable, the set of quotients b/a , and thus the set of rational numbers, is countable. ■

Theorem 20 The set of all real numbers is uncountable.

Proof. Every real number can be represented as a (possibly infinite) sequence of integers (indeed, as a sequence of 0's and 1's in a binary representation). It suffices, then, to show that the set of all sequences whose elements are integers is uncountable. Let this set be called A .

Now, let E be a countable subset of A , and let it contain the sequences s_1, s_2, s_3, \dots . Construct the sequence s as follows: if the n^{th} digit of sequence s_n is a zero, make it a one; if it is nonzero, make it a zero. Then the sequence s differs in at least one digit from every member of E ; $s \notin E$. But clearly $s \in A$, so E is a proper subset of A .

This shows that every countable subset of A is a proper subset of A (since E was arbitrary) - but this in turn means that A itself is uncountable, since if it is not we have shown that A is a proper subset of itself, which is absurd. ■

1.4 Metrics and Norms

Whenever we are talking about a set of objects in mathematics, it is very common that we have a feeling about whether two particular objects are "close" to each other. What we mean is usually that the *distance* between them is small. Although it may be intuitive what the distance between two points is, it is not always that intuitive in a more general setup: for instance, how would you think about the distance between two continuous functions on the unit interval? Between two optimal control problems? Between two economies? Between two preference relations? Here is how we formalize what a distance means:

Definition 21 A *metric space* is a set X , whose elements are called **points**, such that with every two points x and y belonging to X , there is a real number $d(x, y)$ associated with these two points, and called the **distance** from x to y , which satisfies:

- $d(x, y) \geq 0$ (we do not want negative distance),
- $d(x, y) = 0 \iff x = y$ (moreover, we want strictly positive distance between distinct points),
- $d(x, y) = d(y, x)$ (symmetry),
- $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

Example 22 $d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$

Example 23 $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ (this is called *Euclidean distance* - and that is the default in \mathbb{R}^n)

Another related concept is that of a *norm*. Norm is only defined for a linear space; note that we never employed linear structure in defining a metric.

Here we provide a few necessary definitions:

Definition 24 A triple $(V, +, \cdot)$ consisting of a set V , addition $+$:

$$\begin{aligned} V \times V &\rightarrow V \\ (x, y) &\rightarrow x + y \end{aligned}$$

and multiplication \cdot :

$$\begin{aligned} \mathbb{R} \times V &\rightarrow V \\ (\lambda, x) &\rightarrow \lambda \cdot x \end{aligned}$$

is called a **real vector space** if the following 8 conditions hold:

1. $(x + y) + z = x + (y + z)$ for all x, y, z in V .
2. $x + y = y + x$ for all x, y in V .
3. There is an element $0 \in V$ (called the *zero vector*) such that $x + 0 = x$ for any x .
4. For each element $x \in V$, there is an element $-x \in V$ such that $x + (-x) = 0$.
5. $\lambda \cdot (\mu x) = (\lambda\mu) \cdot x$ for all $\lambda, \mu \in \mathbb{R}, x \in V$.
6. $1 \cdot x = x$ for all $x \in V$.
7. $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ for all $\lambda \in \mathbb{R}, x, y \in V$.
8. $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ for all $\lambda, \mu \in \mathbb{R}, x \in V$.

Definition 25 A mapping A of a vector space X into a vector space Y is called a **linear transformation** if

$$A(x_1 + x_2) = Ax_1 + Ax_2 \quad \text{and} \quad A(cx) = cAx$$

for all $x, x_1, x_2 \in X$ and all scalars c .

Definition 26 A *linear space* $L(X, Y)$ is the set of all linear transformations from a vector space X into a vector space Y .

Our definition of the norm will be given for the linear space $L(\mathbb{R}^n, \mathbb{R})$.

Definition 27 A *norm* of a vector is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\forall x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$:

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ iff $x = 0$,
3. $\|\lambda x\| = |\lambda| \|x\|$,
4. $\|x + y\| \leq \|x\| + \|y\|$.

Example 28 $\|x\| = \sqrt{x \cdot x}$, where $x \cdot x$ is an inner product, is a norm (why?)
Note that the inner product of an n -dimensional vector x is simply $\sum_{i=1}^n x_i^2$.

Exercise 29 Show that if $\|x\|$ is a norm, then $d(x, y)$ defined as $d(x, y) = \|x - y\|$ is a metric (it is called the metric, generated by a norm).

Exercise 30 For all four examples of metrics above, determine if there is a norm that generates it.