14.102, Math for Economists
Fall 2004
Lecture Notes, 9/9/2004

These notes are primarily based on those written by George Marios Angeletos for the Harvard Math Camp in 1999 and 2000, and updated by Stavros Panageas for the MIT Math for Economists Course in 2002. I have made only minor changes to the order of presentation, and added some material from Guido Kuersteiner's notes on linear algebra for 14.381 in 2002. The usual disclaimer applies; questions and comments are welcome.

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1 The Basics: Vectors, Matrices, Matrix Operations

1.1 Vectors

An n-dimensional real **vector** $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is an ordered n-tuple of real numbers. As a matter of notational convention, whenever we talk of an n-dimensional vector we mean a **column vector**; that is, a single column of n rows:

$$x = [x_i]_{i=1,...,n} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$$

Letting a prime ($'$) denote the **transpose**, a **row vector** is simply the transpose of a column vector:

$$x' = [x_1 \, x_2 \, ... \, x_n]$$

For the set of all real vectors we may define summation and scalar multiplication, as well as inner product, in the ordinary way. Then, the set of all real vectors $x \in \mathbb{R}^n$ is a linear vector space, and endowed with the Euclidean distance/norm forms our well known **n-dimensional Euclidean vector space**.[1]

Definition 1 *A triple $(V, +, \cdot)$ consisting of a set V, addition $+$:*

$$\begin{array}{ccc} V \times V & \rightarrow & V \\ (x, y) & \rightarrow & x + y \end{array}$$

[1] For more details on vector spaces and Euclidean spaces, see Takayama (1993), ch. 1; Simon and Blume (1994), chs. 10 & 27; or the MathCamp notes by Peter and Leeat. ♠ *ed.note: It is so convinient to transfer responsibility to somebody else! :-)*

and multiplication $\cdot$*:*

$$\begin{array}{rcl} \mathbb{R} \times V & \to & V \\ (\lambda, x) & \to & \lambda \cdot x \end{array}$$

is called a ***real vector space*** *if the following 8 conditions hold:*

1. $(x+y)+z = x+(y+z)$ *for all* x*,* y*,* z *in* V*.*

2. $x+y = y+x$ *for all* x*,* y *in* V*.*

3. There is an element $0 \in V$ *(called the zero vector) such that* $x+0 = x$ *for any* x*.*

4. For each element $x \in V$*, there is an element* $-x \in V$ *such that* $x+(-x) = 0$.

5. $\lambda \cdot (\mu x) = (\lambda\mu) \cdot x$ *for all* λ*,* $\mu \in \mathbb{R}$*,* $x \in V$*.*

6. $1 \cdot x = x$ *for all* $x \in V$*.*

7. $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$ *for all* $\lambda \in \mathbb{R}$*,* x*,* $y \in V$*.*

8. $(\lambda+\mu) \cdot x = \lambda \cdot x + \mu \cdot x$ *for all* λ*,* $\mu \in \mathbb{R}$*,* $x \in V$*.*

1.2 Matrices

A real **matrix** is defined as a rectangular array of real numbers.[2] In particular, the matrix

$$A = [a_{ij}]_{i=1,\ldots,m}^{j=1,\ldots,n} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

for $a_{ij} \in \mathbb{R}\ \forall i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$ and $m, n \in \mathbb{N}_* = \{1, 2, ...\}$, is a matrix of dimensions $m \times n$. More compactly we write this matrix as $A = [a_{ij}]$.

Notice that, just as an n-dimensional (real) vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is an ordered set or ordered n-tuple of real numbers, an $m \times n$ matrix is an ordered set or ordered n-tuple of m-dimensional vectors. Thus, we may write a matrix as an array of column vectors

$$A = [a_j] = [a_1\, a_2\, ...\, a_n]$$

with the understanding that $a_j = (a_{1j}, a_{2j}, ..., a_{mj}) \in \mathbb{R}^m$ is an m-dimensional column vector for all $j = 1, .., n$.

[2] In most of our discussion we restrict focus to matrices over the set of real numbers, $\mathbb{R}$. All notions, however, can be extended to the complex plane, $\mathbb{C}$, or any arbitrary field. It's a good exercise for the reader to check what (if any) would have to change should, throughout this text, we had set $\mathbb{C}$ wherever $\mathbb{R}$ appears.

1.3 Matrix Transpose

The **transpose** A' of an $m \times n$ matrix $A = [a_{ij}]$ is an $n \times m$ matrix defined simply as

$$A' = [a_{ji}] = \begin{bmatrix} a_1' \\ a_2' \\ \dots \\ a_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

1.4 Matrix Addition and Scalar Multiplication

The set of all matrices of the same dimensions forms a vector space. This requires only that we appropriately define matrix **addition** and **scalar multiplication**:

Definition 2 *Take any $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, and any real scalars λ, μ. We define **matrix addition** as*

$$C = A + B \;\; \Leftrightarrow \;\; c_{ij} = a_{ij} + b_{ij} \;\; \forall i, j$$

*and **scalar multiplication** as*

$$C = \lambda A \;\; \Leftrightarrow \;\; c_{ij} = \lambda a_{ij} \;\; \forall i, j$$

We can then easily show the following properties:

Lemma 3 *Matrix addition and scalar multiplication, defined as above, satisfy:*

(i) *Commutative Law:* $A + B = B + A$
(ii) *Associative Law:* $(A + B) + C = A + (B + C)$
(iii) *Distributive Law:* $(\lambda + \mu)A = \lambda A + \mu A$
(iv) *Distributive Law:* $\lambda(A + B) = \lambda A + \lambda B$
(v) *Distributive Law:* $\lambda(\mu A) = (\lambda\mu)A$

It then follows that:

Proposition 4 *The set of all $m \times n$ real matrices endowed with matrix addition and scalar multiplication forms a linear vector space.*

Exercise 5 *Persuade yourself that you can prove Lemma 3 and Proposition 4 from first principles.*

1.5 Matrix Multiplication

Let A and B be matrices. The matrix product of A and B (denoted AB) is defined when A is $m \times n$ and B is $n \times p$. When this holds (i.e., there are as many columns in A as there are rows in B), we say that A and B are *conformable*. The i, k^{th} element of AB is then given by

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

and AB is an $m \times p$ matrix.

Matrix multiplication satsfies the following properties:

Lemma 6 *Provided dimension conformity, matrix multiplication satisfies:*

$$\begin{array}{l} (i)\ \textit{Associative Law:}\ \ (AB)C = A(BC) \\ (ii)\textit{Associative Law:}\ \ (\lambda A)B = \lambda(AB) \\ (iii)\ \textit{Distributive Law:}\ \ A(B+C) = AB + AC \\ (iv)\ \textit{Distributive Law:}\ \ (A+B)C = AC + BC \end{array}$$

Note that if AB exists, BA need not be definted; even if it does exist, it is not generally true that $AB = BA$ (when this is true, we say that A and B are *commuting*). This is why we need both (iii) and (iv) above; the first distributive law deals with pre-multiplication of a matrix sum by another matrix, while the second deals with post-multiplication of a matrix sum with another matrix.

1.6 Special Matrices

Definition 7 *An $m \times n$ matrix A is called* ***square*** *if $m = n$.*

A matrix A is ***symmetric*** *if $A' = A$, or equivalently $a_{ij} = a_{ji}\ \forall i, j$. This imposes that A is square.*

A matrix A is ***upper triangular*** *if it is square and $a_{ij} = 0$ for $j = i+1, ..., n$ and $i = 1, ..., n-1$. And A is* ***lower triangular*** *iff A' is upper triangular.*

A matrix A is ***diagonal*** *if it is both upper and lower triangular; equivalently, iff $a_{ij} = 0$ for $i \neq j$.*

The ***null matrix*** *(denoted 0) is defined by the relation:*

$$A + (-1)A = 0, \text{ or equivalently, } A + 0 = A$$

It can thus take any dimension (it must have the same dimension as A), and is the matrix with all elements equal to zero.

The ***identity matrix*** *(denoted I) is defined by the relation:*

$$IA = AI = A$$

It is the diagonal matrix with ones on its diagonal, and takes the dimension necessary for conformability (that is, if A above is $m \times n$, I is $m \times m$ in the first part of the above relation and $n \times n$ in the second part).

1.7 Transpose Rules

Lemma 8 *The following rules hold for matrix transposes:*

$$\begin{aligned} (A')' &= A \\ (A+B)' &= A' + B' \\ (AB)' &= B'A' \\ A'A &\neq AA \neq A'A' \text{ in general, but} \\ A'A &= AA = A'A' \text{ for symmetric } A \end{aligned}$$

Note that AB defined $\Rightarrow (AB)' = B'A'$ defined.

Exercise 9 *Find examples illustrating the above facts.*