14.102, Math for Economists
Fall 2005
Lecture Notes, 9/13/2005

These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples, mostly from Rudin. The usual disclaimer applies; questions and comments are welcome.

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2 Real Analysis II - Sets and Functions

2.1 Sequences and Limits

The concept of a sequence is very intuitive - just an infinite ordered array of real numbers (or, more generally, points in $\mathbb{R}^n$) - but is defined in a way that (at least to me) conceals this intuition.

One point to make here is that a sequence in mathematics is something *infinite*. In our everyday language, instead, we sometimes use the word "sequence" to describe something finite (like "sequence of events", for example).

Definition 31 *A (finite) number A is called the limit of sequence $\{a_n\}$ if $\forall \varepsilon > 0$ $\exists N : \forall n > N \ \ |a_n - A| < \varepsilon$. If such number A exists, the sequence is said to be convergent.*

Verbally, A is the limit of $\{a_n\}$ if the sequence comes closer and closer to A as N grows and, moreover, *stays* close to A "forever". Of course, such A does not have to exist, as the following simple example shows:

Example 32 *Let $a_n = (-1)^n$. Then $\{a_n\}$ does not have any limit.*

That is, a sequence does not have to converge to any single point (for example, it can oscillate between two different points). However, what it surely can never do is to converge to two distinct points at a time:

Lemma 33 *A sequence can have at most one limit.*

Definition 34 *Sequence $\{a_n\}$ is said to converge to ∞ (with no sign) if $\forall C$ $\exists N : \forall n > N \ \ |a_n| > C$.*

Our next step is to capture the fact that even a divergent (i.e., nonconvergent) sequence can still have "frequently visited" or "concentration" points - points to which infinitely many terms of the sequence are "close". This intuition is captured in

Definition 35 *A (finite) number B is called a limit point of* $\{a_n\}$ *if* $\forall\varepsilon > 0$ $\forall N \exists n > N: \ |a_n - B| < \varepsilon$.

Exercise 36 *Find all limit points of the sequence in the sequence* $\{a_n\}$ *with* $a_n = (-1)^n$. *Recall that this sequence has no* ***limit****..*

Exercise 37 *The only limit point of a convergent sequence is its limit.*

Example 38 *The converse does not hold: consider sequence* $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}$, *etc. Its only limit point is 0 (why?) but it does not converge to it.*

Exercise 39 *Define the meaning of* ∞ *being a limit point of a sequence.*

To be convergent is a strong condition on $\{a_n\}$; to have a limit point is a weaker condition. The price you have to pay for relaxing this (or any) condition is that now more points will fit - for example, a sequence can have only one limit (which adds some desired definitiveness to the concept) but multiple limit points. What you hope to get in return is that more sequences have limit points than have limits[1]. To make an exact statement we need one more

Definition 40 *Sequence* $\{a_n\}$ *is called bounded if* $\exists C : \forall n \ \ |a_n| < C$

Exercise 41 *Every convergent sequence is bounded.*

Now we are ready for

Theorem 42 *(Bolzano-Weierstraß) Every bounded sequence has a limit point.*

This theorem is often stated as 'Every bounded sequence has a convergent subsequence.' The idea is simple: if a sequence has a limit point, then we know that no matter how far out into the sequence we get, we always return to an arbitrarily small neighborhood of the limit point eventually (before possibly leaving again, and returning, and leaving...). So we can construct an infinite subsequence, selecting only the points sufficiently close to the limit point, which in fact *converges* to the limit point as its *limit.*

2.2 Cauchy Sequences

The 'Cauchy method' is often useful in establishing the convergence of a given sequence, without necessarily defining the limit to which it converges.

Definition 43 *A sequence* $\{p_n\}$ *in a metric space X is said to be a* ***Cauchy sequence*** *if for every* $\varepsilon > 0$ *there is an integer N such that* $d(p_n, p_m) < \epsilon$ *if* $n \geq N$ *and* $m \geq N$.

[1] A similar tradeoff arises in game theory: we can use strictly dominant strategies or Nash equilibrium as a solution concept; the former is more definite and probably more appealing, but need not (and in most interesting cases does not) exist; the latter always exists (for finite games at least) but need not be unique and deserves further justification. Now that, after a number of years in economics, I have finally learned the fundamental concept of tradeoff, I am amazed to see in how many instances it is applicable in math.

Theorem 44 *1. In any metric space X, every convergent sequence is a Cauchy sequence.*

2. In $\mathbb{R}^k$, every Cauchy sequence converges.

Proof.

1. *If $p_n \to p$ and if $\epsilon > 0$, there is an integer N such that $d(p, p_n) < \epsilon$ for all $n \geq N$. Hence*

$$d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$$

as long as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

2. *(Sketch) The full proof requires concepts that we won't go into, but it can be found in Rudin, Theorem 3.11. The idea of the proof is that if $\{p_n\}$ is Cauchy, then we know that there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$. Let $n \geq N$, for the N in the hypothesis. Then $d(p_n, p_{n+1}) < \epsilon$, $d(p_{n+1}, p_{n+2}) < \epsilon, ..., d(p_{n+i}, p_{n+i+1}) < \epsilon$, for all $i \geq 0$. By the triangle inequality, $d(p_n, p_{n+i+1}) \leq \sum_{j=0}^{i} d(p_{n+i}, p_{n+i+1}) < i\varepsilon$. So the sequence is 'converging' to a 'limit' of p_n. Of course, the 'sketchiness' of this proof arises in the fact that p_n is not necessarily the limit of the sequence, and that we run into trouble with this argument when we let n go to infinity - but this gives about the right intuition.*

■

2.3 Open and Closed Sets

For the rest of the analysis we stick to the Euclidean metric on $\mathbb{R}^n$: $d(x, y) = d_2(x, y)$.

Definition 45 *For any $x_0 \in \mathbb{R}^n$ and $r > 0$ define an open ball $B_r(x_0) = \{x \in \mathbb{R}^n | d(x, x_0) < r\}$.*

Exercise 46 *What do open balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like? What would they look like if we fixed another metric (d_1 or d_∞) instead of d_2?*

Definition 47 *Set $A \subset \mathbb{R}^n$ is called **open** if, together with any point $x_0 \in A$, it contains a small enough open ball $B_\varepsilon(x_0)$ for some $\varepsilon > 0$.*

Example 48 *An open ball is an open set (why?)*

Example 49 *The half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ is open*

Exercise 50 *The union of any (not necessarily finite) number of open sets is open; the intersection of two (or any finite number of) open sets is open.*

Example 51 *Let $A_n = \{-\frac{1}{n} < x < \frac{1}{n}\}$. Persuade yourself that A_n is open for all n. What is the intersection of all $A_n, n = 1, 2, ...$? Show that it is not open.*

Definition 52 *A point p is a* ***limit point*** *of a set C if every open ball centered at p contains a point* $q \neq p$ *such that* $q \in C$.

Definition 53 *Set C is called* ***closed*** *if it contains all its limit points.*

Lemma 54 *A set C is closed if and only if its complement is open.*
Proof. *First, suppose* C^c *is open. Let x be a limit point of C. Then every open ball centered at x contains a point of C, so that x is not an interior point of* C^c. *Since* C^c *is open, this means that* $x \in C$. *It follows that C is closed.*

Second, suppose that C is closed. Choose $x \in C^c$. *Then* $x \notin C$, *and x is not a limit point of C. Hence there exists an open ball* $B_r(x)$ *such that* $C \cap B_r(x)$ *is empty, which implies* $B_r(x) \subset C^c$. *Thus, x is an interior point of* C^c*; being true for all* $x \in C^c$, *this means that* C^c *is open.* ■

Example 55 *A closed ball* $B_r(x_0) = \{x \in \mathbb{R}^n | d(x, x_0) \leq r\}$ *is a closed set.*

Definition 56 *If X is a metric space, if* $E \subset X$, *and if* E' *denotes the set of all limit points of E in X, then the* ***closure*** *of E is the set* $E \cup E'$.

Exercise 57 *Show that empty set* $\emptyset$ *and the entire space* $\mathbb{R}^n$ *are both open and closed. Persuade yourself that these two are the only sets which are both open and closed.*

Definition 58 *A set in* $\mathbb{R}^n$ *is called* ***compact*** *if it is closed and bounded.*

This is not the traditional definition of compactness that you will find in a textbook – in spaces more general than $\mathbb{R}^n$ it will not work (that is, in those spaces there exist closed and bounded sets which will not be compact). However, in $\mathbb{R}^n$ it will work fine: whatever definition of compactness you will ever see, it will be equivalent to the one above.

2.4 Convexity (sets) and Separating Hyperplanes

There is a branch of real analysis which plays a relatively modest role in pure mathematics, but is an enormously powerful device in economics. It has to do with the notion of convexity.

Unlike topological concepts such as open, closed and compact sets (which in principal require very little structure on the space), convexity makes use of a linear structure.

Definition 59 *A convex combination of points x and y in* $\mathbb{R}^n$ *is any point z that can be expressed as* $z = \alpha x + (1 - \alpha)y$ *for some real number* $\alpha \in [0, 1]$.

The set of all convex combinations of two given points is the closed segment between them.

Definition 60 *A set* $A \subset \mathbb{R}^n$ *is called convex if, together with any two points* $x, y \in A$ *it contains all their convex combinations.*

Exercise 61 *Show that an intersection of (even infinitely many) convex sets is convex.*

Definition 62 *The convex hull, denoted conv*(A)*, of set* A *is the intersection of all convex sets that contain* A*. It is the smallest convex set containing* A[2]

Example 63 *An open (or closed) ball is a convex set.*

Example 64 *The half-space is a convex set.*

Definition 65 *Let* $p \neq 0$ *be a vector in* $\mathbb{R}^n$*,and let* $a \in \mathbb{R}$*. The set* H *defined by* $H = \{x \in \mathbb{R}^n | p \cdot x = a\}$ *is called a hyperplane in* $\mathbb{R}^n$*. We denote it by* $H(p, a)$*.*

Hyperplanes in $\mathbb{R}^2$ are straight lines, hyperplanes in $\mathbb{R}^3$ are usual planes and, generally, hyperplanes in $\mathbb{R}^n$ are spaces of dimension $n - 1$.

The key result (which is indispensable for the second welfare theorem and a variety of other economic results) is the following

Theorem 66 *(Separating Hyperplane Theorem) Let* C *be a nonempty convex set in* $\mathbb{R}^n$ *and let* x^* *be a point in* $\mathbb{R}^n$ *that is not in* C*. Then there exists a hyperplane* $H(p, a)$ *that separates* C *and* x^**, i.e., such that* $p \cdot y \leq a$ *for all* $y \in C$ *and* $p \cdot x^* \geq a$*.*

Exercise 67 *Nonstrict inequalities (*$\leq$ *and* $\geq$*) are essential and can not, in general, be replaced by strict inequalities (*$<$ *and* $>$*). Construct an example of a convex set and a point outside it that can not be strictly separated.*

A slightly more general result is

Theorem 68 *Let* C_1 *and* C_2 *be two disjoint (i.e.,* $C_1 \cap C_2 = \emptyset$*) convex sets in* $\mathbb{R}^n$*. Then there exists a hyperplane* $H(p, a)$ *that separates* C_1*and* C_2*, i.e., such that* $\forall x \in C_1 \;\; p \cdot x \leq a$ *and* $\forall y \in C_2 \;\; p \cdot y \geq a$*.*

Example 69 *Any point on the contract curve in the Edgeworth box is a Walrasian equilibrium with an appropriate price vector, as soon as preferences are concave.*

Example 70 *The optimal (from the central planning standpoint) production/consumption choice in a Robinson Crusoe economy can be supported as a decentralized equilibrium, as long as the production possibility set is convex and preferences are concave.*

[2] Likewise, since the intersection of any number of closed sets is closed, we can define the *closure* of set A as the intersection of all closed sets containing A, which will then be the smallest closed superset of A. However, it is straightforward to see that, in general, there will be no such thing as the smallest *open* set containg A (think, for example, of $A = \{0\}$).

2.5 Functions

Typically mathematicians rely on the notion of a *function* as being built in to general perception of abstract mathematical objects, which means that they do not bother to rigorously define it. However, given Kantor's program of redefining everything in terms of sets only, functions also needed to be so defined. And the way Kantor did it was simple: he just identified a function with its graph. Namely, a *function* (for example, from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a subset $graph(f)$ of the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ with one restriction that any $x \in \mathbb{R}^n$ may be mapped only to one $y \in \mathbb{R}^m$,i.e., that $(x, y_1) \in F \& (x, y_2) \in F \implies y_1 = y_2$. This last property allows us to define $y = y(x)$ unambiguously.

Note that according to the definition above a function does not have to map the entire space $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., it does not have to be the case that $\forall x \in \mathbb{R}^n$ $\exists y \in \mathbb{R}^m$ such that $(x, y) \in graph(f)$. If such y does exist for a given x (in which case it is unique by definition), we say that x belongs to the *domain* of f.

Exercise 71 *Find the domains of the following functions* $f : \mathbb{R} \to \mathbb{R}$:

1. $f(x) = \sqrt{x}$
2. $f(x) = \frac{1}{x^2+2x-3}$
3. $f(x) = \frac{1}{\sin x} + \frac{1}{\cos x}$

A subset of $\mathbb{R}^n \times \mathbb{R}^m$ can easily be converted to a subset of $\mathbb{R}^m \times \mathbb{R}^n$ by switching coordinates. The resulting subset does not have to be a function (in the sense that there does not have to be unique x associated with each y), but if it is a function it is called the *inverse* function and is denoted $f^{-1}(y)$.

The above definitions are quite abstract, but the main idea of a function is probably clear to everybody.

The concept of a function (usually in terms of one variable being uniquely determined by another, in our case y being determined by x) has been understood for a very long time, by the ancient Greeks at least (although they did not quite operate in terms of *variables*, let alone using letters to denote variables, which was introduced by a French mathematician Vieta in late XVI century for purposes entirely different from those we use them for nowadays). The birth of modern calculus is dated back to Newton and Leibniz I think (in the mid XVII century) and the fundamental notion they managed to give a formal definition for was *continuity.* Informally, a continuous function is a function, whose graph we can draw in one touch of a pencil, without taking it off the paper. Although intuitively quite appealing, this "definition" is not operative, so we need a formal (and, of course much less intuitive) one:

Definition 72 *A function* $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ *is said to be continuous at point* $x_0 \in X$ *if* $\forall \varepsilon > 0 \exists \delta > 0$ *such that* $\forall x \in B_\delta(x_0)$, $f(x) \in B_\varepsilon(f(x_0)) \subset \mathbb{R}^m$. *A function is said to be continuous on* X *if it is continuous at each point of* X.

Example 73 $f(x) = sign(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ *is continuous at all points except 0.*

Example 74 $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ *is not continuous anywhere (it is called Dirichlet function after another French mathematician Lejeune Dirichlet, 1805-1859)*

Example 75 $g(x) = xD(x)$*, where* $D(x)$ *is Dirichlet function defined above, is continuous at* $x_0 = 0$ *and nowhere else.*

Example 76 $R(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ with minimum possible } p \text{ and } q \\ 0, & x \notin \mathbb{Q} \end{cases}$ *is continuous at all irrational points* $x \notin \mathbb{Q}$ *and not continuous at rational points (it is called Riemann function after a German mathematician Georg Riemann, 1826-1866).*

Exercise 77 *Make sure that you fully understand the examples above.*

Example 78 *Usual functions are continuous on their domains: ln, exp, sin, cos, polynomials, radicals.*

Lemma 79 *A sum, a difference and a product of two continuous functions are continuous; a ratio of two continuous functions is continuous at all points where the denominator does not vanish.*

Let f be a strictly increasing continuous function on $[a, b]$ then $f([a, b]) = [f(a), f(b)]$ and f is a one-to-one function from $[a, b]$ onto $[f(a), f(b)]$.

Then the inverse f^{-1} is a one-to-one function from $[f(a), f(b)]$ onto $[a, b]$.

Theorem 80 *If* f *is continuous and increasing then so is* f^{-1}*.*

Draw graph and note that f and f^{-1} are symmetric with respect to the 45^0 line.

Definition 81 *A function on* $I = [a, b]$ *bounded interval is* ***piece-wise continuous*** *if it is continuous everywhere except on a finite number of points in* I *and that at every point where it is not continuous it admits left and right limits.*

2.6 Fixed-Point Theorems

Theorem 82 ***Intermediate Value Theorem*** *Let* f *be a continuous function on a closed interval* $[a, b]$*. Then for any* m *in the interval* $[f(a), f(b)]$ *or* $([f(b), f(a)])$*, there is some* $c \in [a, b]$ *such that* $f(c) = m$*.*

Proof. Suppose $f(a) < f(b)$. Let $S = \{x \in [a,b] | f(x) \leq m\}$, S non empty and bounded. So consider $c = \sup S$. Take x_n converging to c, of course $f(x_n) \leq m$ so $f(c) \leq m$. Now take $x_n > c$ converging to c. $f(x_n) > m$ so the limit $f(c) \geq m$. So $f(c) = m$. ■

Example: existence of an equilibrium price. If $Supply(0) = Demand(+\infty) = 0$ and $Supply(+\infty) = Demand(0) = +\infty$ then $\exists p$ such that $Supply(p) = Demand(p)$.

This example is related to fixed point theorems:

Theorem 83 *(Brouwer's) Let A be a convex and compact subset of R (or R^n) and let $f : A \to A$ be a continous function. Then, there exists a fixed point of f that is a point $x \in A$ such that*

$$f(x) = x$$

A set $A \subseteq R^n$ is called convex if $a, b \in A$ implies that the entire segment between a and b is contained in A.

Theorem 84 *(Tarski's) Let A be a set with ordering $\leq$ such that each subset of A has a* sup *and an* inf. *Let $f : A \to A$ be monotonic (weakly increasing or decreasing). Then, f has a fixed point.*

Theorem 85 *(Banach's) Let A be a space with metrics $d(\cdot, \cdot)$ and $f : A \to A$ be a contraction that is there exists $\lambda \in (0,1)$ such that*

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Then, f has a unique fixed point.

A version of the Brouwer theorem (Kakutani's) is used to prove the existence of Nash equilibria in many games. Tarski's theorem is used to prove existence of equilibria of supermodular games. Banach's theorem is the workhorse of the theory of differential equations that will be studied in 14.102.