14.102, Math for Economists
Fall 2005
Lecture Notes, 9/15/2005

These notes are primarily based on those written by George Marios Angeletos for the Harvard Math Camp in 1999 and 2000, and updated by Stavros Panageas for the MIT Math for Economists Course in 2002. I have made only minor changes to the order of presentation, and added some material from Guido Kuersteiner's notes on linear algebra for 14.381 in 2002. The usual disclaimer applies; questions and comments are welcome.

Nathan Barczi
nab@mit.edu

2.7 Continuity and Upper/Lower Hemicontinuity

Kakutani's fixed point theorem weakens the conditions of Brouwer's theorem so that it applies to more games - indeed, to all finite strategic-form games. 'Finite' refers to the number of players and the actions they have to choose from; Glenn will go over this, as well as the distinction between strategic-form and extensive-form games, in more detail. He will also discuss how such games are interpreted to fit the conditions of the theorem. For now, our concern is to achieve an understanding of those conditions.

Kakutani's theorem is as follows:

Theorem 86 *(Kakutani) Let Σ be a compact, convex, nonempty subset of a finite-dimensional Euclidean space, and $r : \Sigma \rightrightarrows \Sigma$ a correspondence from Σ to Σ which satisfies the following:*

1. *$r(\sigma)$ is nonempty for all $\sigma \in \Sigma$.*
2. *$r(\sigma)$ is convex for all $\sigma \in \Sigma$.*
3. *$r(\cdot)$ has a closed graph.*

Then r has a fixed point.

Everything in this theorem is familiar from our previous discussion, with the exception of the third requirement for r, that it have a closed graph. This property is also referred to as upper-hemi continuity.

Definition 87 *A compact-valued correspondence $g : A \rightrightarrows B$ is **upper hemicontinuous** at a if $g(a)$ is nonempty and if, for every sequence $a_n \to a$ and every sequence $\{b_n\}$ such that $b_n \in g(a_n)$ for all n, there exists a convergent subsequence of $\{b_n\}$ whose limit point b is in $g(a)$.*

In words, this says that for every sequence of points in the graph of the correspondence that converges to some limit, that limit is also in the graph of the correspondence. This means that we don't 'lose points' in our graph at the limit of a convergent sequence of points in the graph, and important property for ensuring that we have a fixed point.

There is also a property called lower hemi-continuity:

Definition 88 *A correspondence $g : A \rightrightarrows B$ is said to be **lower hemi-continuous** at a if $g(a)$ is nonempty and if, for every $b \in g(a)$ and every sequence $a_n \to a$, there exists $N \geq 1$ and a sequence $\{b_n\}_{n=N}^{\infty}$ such that $b_n \to b$ and $b_n \in g(a_n)$ for all $n \geq N$.*

In words, this says that for every point in the graph of the correspondence, if there is a sequence in A converging to a point a for which $g(a)$ is nonempty, then there is also a sequence in B converging to $b \in g(a)$, and that every point b_n in that sequence is in the graph of a_n.

Together, these two give us continuity:

Definition 89 *A correspondence $g : A \rightrightarrows B$ is **continuous** at $a \in A$ if it is both u.h.c and l.h.c. at a.*

2.8 Convexity (functions)

Definition 90 *Let $f : I \longrightarrow R$ be a function on the interval I. f is convex iff for any x and y in I and any $0 < \lambda < 1$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

f is concave iff for any x and y in I and any $0 < \lambda < 1$

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

Some seful facts:
- f is concave iff $-f$ is convex (and similarly for strictly concave/convex)
- the sum of convex functions is convex.
- the sum of concave functions is concave.

Graphically, a convex (concave) function is below (above) the segment $[(x, f(x)), (y, f(y))]$.

Convexity guarantees uniqueness of a maximum (provided a maximum exists). Convexity of a continuous function on a compact domain in R^n guarantees (a) existence of the maximum and (b) uniqueness of the maximum. This is often used in maximization problems.

2.9 Quasiconvexity and Quasiconcavity

One problem with concavity and convexity (which we'll encounter again when we look at homogeneity) is that they are *cardinal* properties. That is, whether or not a function is concave depends on the numbers which the function assigns to its level curves, not just to their shape. The problem with this is that a monotonic transformation of a concave (or convex) function need not be concave (or convex). For example, $f(x) = \frac{-x^2}{2}$ is concave, and $g(x) = e^x$ is a monotonic transformation, but $g(f(x)) = e^{-\frac{x^2}{2}}$ is not concave. This is problematic when we want to analyze things like utility which we consider to be ordinal concepts.

A weaker condition to describe a function is quasiconvexity (or quasiconcavity). Functions which are quasiconvex maintain this quality under monotonic transformations; moreover, every monotonic transformation of a concave function is quasiconcave (although it is not true that every quasiconcave function can be written as a monotonic transformation of a concave function).

Definition 91 *A function f defined on a convex subset U of $\mathbb{R}^n$ is **quasiconcave** if for every real number a,*

$$C_a^+ \equiv \{x \in U : f(x) \geq a\}$$

*is a convex set. Similarly, f is **quasiconvex** if for every real a,*

$$C_a^- \equiv \{x \in U : f(x) \leq a\}$$

is a convex set.

The following theorem gives some equivalent definitions for quasiconcavity:

Theorem 92 *Let f be a function defined on a convex subset U in $\mathbb{R}^n$. Then the following statements are equivalent:*
(a) f is a quasiconcave function on U.
(b) For all $x, y \in U$ and all $t \in [0,1]$,

$$f(x) \geq f(y) \text{ implies } f(tx + (1-t)y) \geq f(y)$$

(c) For all $x, y \in U$ and all $t \in [0,1]$,

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

Exercise 93 *For a function f defined on a convex subset U in $\mathbb{R}^n$, show that f concave implies f quasiconcave.*

The previous exercise shows what we mean when we say that quasiconcavity is weaker than concavity. Moreover, as noted previously, monotone transformations of quasiconcave functions remain quasiconcave, allowing us to use them to represent ordinal concepts such as utility. From our point of view, looking at optimization, the important point is that a critical point of many quasiconcave functions will be a maximum, just as is the case with a concave function. But such critical points need not exist - and even if they do, they are not necessarily maximizers of the function - consider $f(x) = x^3$. Any strictly increasing function is quasiconcave *and* quasiconvex (check this); this function is both over the compact interval $[-1, 1]$, but the critical point $x = 0$ is clearly neither a maximum nor a minimum over that interval. What we usually use these concepts for is to check that upper contour sets (which can represent demand correspondences, or sets of optimal strategies in game theory, etc.) are convex.

3 Vectors, Matrices, Matrix Operations

3.1 Vectors

An n-dimensional real **vector** $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is an ordered n-tuple of real numbers. As a matter of notational convention, whenever we talk of an n-dimensional vector we mean a **column vector**; that is, a single column of n rows:

$$x = [x_i]_{i=1,...,n} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$$

Letting a prime ($'$) denote the **transpose**, a **row vector** is simply the transpose of a column vector:

$$x' = [x_1 \, x_2 \, ... \, x_n]$$

For the set of all real vectors we may define summation and scalar multiplication, as well as inner product, in the ordinary way. Then, the set of all real vectors $x \in \mathbb{R}^n$ is a linear vector space, and endowed with the Euclidean distance/norm forms our well known **n-dimensional Euclidean vector space**.[3]

3.2 Matrices

A real **matrix** is defined as a rectangular array of real numbers.[4] In particular, the matrix

$$A = [a_{ij}]_{i=1,...,m}^{j=1,...,n} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

for $a_{ij} \in \mathbb{R} \; \forall i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$ and $m, n \in \mathbb{N}_* = \{1, 2, ...\}$, is a matrix of dimensions $m \times n$. More compactly we write this matrix as $A = [a_{ij}]$.

Notice that, just as an n-dimensional (real) vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is an ordered set or ordered n-tuple of real numbers, an $m \times n$ matrix is an ordered set or ordered n-tuple of m-dimensional vectors. Thus, we may write a matrix as an array of column vectors

$$A = [a_j] = [a_1 \, a_2 \, ... \, a_n]$$

with the understanding that $a_j = (a_{1j}, a_{2j}, ..., a_{mj}) \in \mathbb{R}^m$ is an m-dimensional column vector for all $j = 1, .., n$.

---

[3] For more details on vector spaces and Euclidean spaces, see Takayama (1993), ch. 1; Simon and Blume (1994), chs. 10 & 27; or the MathCamp notes by Peter and Leeat. ♠ *ed.note: It is so convinient to transfer responsibility to somebody else! :-)*

[4] In most of our discussion we restrict focus to matrices over the set of real numbers, $\mathbb{R}$. All notions, however, can be extended to the complex plane, $\mathbb{C}$, or any arbitrary field. It's a good exercise for the reader to check what (if any) would have to change should, throughout this text, we had set $\mathbb{C}$ wherever $\mathbb{R}$ appears.

3.3 Matrix Transpose

The **transpose** A' of an $m \times n$ matrix $A = [a_{ij}]$ is an $n \times m$ matrix defined simply as

$$A' = [a_{ji}] = \begin{bmatrix} a_1' \\ a_2' \\ ... \\ a_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ ... & ... & & ... \\ a_{1n} & a_{2n} & ... & a_{mn} \end{bmatrix}$$

3.4 Matrix Addition and Scalar Multiplication

The set of all matrices of the same dimensions forms a vector space. This requires only that we appropriately define matrix **addition** and **scalar multiplication**:

Definition 94 *Take any $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, and any real scalars λ, μ. We define **matrix addition** as*

$$C = A + B \;\; \Leftrightarrow \;\; c_{ij} = a_{ij} + b_{ij} \;\; \forall i, j$$

*and **scalar multiplication** as*

$$C = \lambda A \;\; \Leftrightarrow \;\; c_{ij} = \lambda a_{ij} \;\; \forall i, j$$

We can then easily show the following properties:

Lemma 95 *Matrix addition and scalar multiplication, defined as above, satisfy:*

(i) *Commutative Law:* $A + B = B + A$
(ii) *Associative Law:* $(A + B) + C = A + (B + C)$
(iii) *Distributive Law:* $(\lambda + \mu)A = \lambda A + \mu A$
(iv) *Distributive Law:* $\lambda(A + B) = \lambda A + \lambda B$
(v) *Distributive Law:* $\lambda(\mu A) = (\lambda\mu)A$

It then follows that:

Proposition 96 *The set of all $m \times n$ real matrices endowed with matrix addition and scalar multiplication forms a linear vector space.*

Exercise 97 *Persuade yourself that you can prove Lemma 95 and Proposition 96 from first principles.*

3.5 Matrix Multiplication

Let A and B be matrices. The matrix product of A and B (denoted AB) is defined when A is $m \times n$ and B is $n \times p$. When this holds (i.e., there are as many columns in A as there are rows in B), we say that A and B are *conformable.* The i, k^{th} element of AB is then given by

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

and AB is an $m \times p$ matrix.

Matrix multiplication satsfies the following properties:

Lemma 98 *Provided dimension conformity, matrix multiplication satisfies:*

$$\begin{array}{l} (i)\ \textit{Associative Law:}\ \ (AB)C = A(BC) \\ (ii)\textit{Associative Law:}\ \ (\lambda A)B = \lambda(AB) \\ (iii)\ \textit{Distributive Law:}\ \ A(B+C) = AB + AC \\ (iv)\ \textit{Distributive Law:}\ \ (A+B)C = AC + BC \end{array}$$

Note that if AB exists, BA need not be definted; even if it does exist, it is not generally true that $AB = BA$ (when this is true, we say that A and B are *commuting*). This is why we need both (iii) and (iv) above; the first distributive law deals with pre-multiplication of a matrix sum by another matrix, while the second deals with post-multiplication of a matrix sum with another matrix.

3.6 Special Matrices

Definition 99 *An $m \times n$ matrix A is called* ***square*** *if $m = n$.*

A matrix A is ***symmetric*** *if $A' = A$, or equivalently $a_{ij} = a_{ji}$ $\forall i, j$. This imposes that A is square.*

A matrix A is ***upper triangular*** *if it is square and $a_{ij} = 0$ for $j = i+1, ..., n$ and $i = 1, ..., n-1$. And A is* ***lower triangular*** *iff A' is upper triangular.*

A matrix A is ***diagonal*** *if it is both upper and lower triangular; equivalently, iff $a_{ij} = 0$ for $i \neq j$.*

The ***null matrix*** *(denoted 0) is defined by the relation:*

$$A + (-1)A = 0, \textit{ or equivalently, } A + 0 = A$$

It can thus take any dimension (it must have the same dimension as A), and is the matrix with all elements equal to zero.

The ***identity matrix*** *(denoted I) is defined by the relation:*

$$IA = AI = A$$

It is the diagonal matrix with ones on its diagonal, and takes the dimension necessary for conformability (that is, if A above is $m \times n$, I is $m \times m$ in the first part of the above relation and $n \times n$ in the second part).

3.7 Transpose Rules

Lemma 100 *The following rules hold for matrix transposes:*

$$\begin{array}{rcl} (A')' & = & A \\ (A+B)' & = & A' + B' \\ (AB)' & = & B'A' \\ A'A & = & 0 \Leftrightarrow A = A' = 0 \\ A'A & \neq & AA \neq A'A' \textit{ in general, but} \\ A'A & = & AA = A'A' \textit{ for symmetric } A \end{array}$$

Note that AB defined $\Rightarrow (AB)' = B'A'$ defined.

Exercise 101 *Find examples illustrating the above facts.*