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3 Systems of Equations

3.1 Linear Systems

A **linear system** of m equations in n unknowns (or, for brevity, an $m \times n$ linear system) is generally written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where the a_{ij}'s and b_i's ($i = 1, ..., m$ and $j = 1, ..., n$) are coefficients and constants taken from $\mathbb{R}$ and the x_j's are the unknowns. A **solution** to (1) is any n-tuple of real numbers x_j's that satisfy simultaneously all m equations in (1).

We may write the above system in a more compact format if we let A be an $m \times n$ matrix formed by the coefficients a_{ij}'s, x the $n \times 1$ column vector of the unknowns x_j's, and b the $m \times 1$ column vector of constants b_i's:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then we rewrite (1) simply as

$$Ax = b \tag{2}$$

We call this an $m \times n$ system, and we call it **square** iff $m = n$.

The **set of solutions** for (2) is simply the set

$$X^* \equiv X^*(A, b) \equiv \{x \in \mathbb{R}^n | Ax = b\}$$

In general, for given A and b, X^* may be empty (**no solution**), or be a singleton (**unique solution**), or have more than one elements (**multiple solutions**). Our task is now to identify conditions on the given A and b that give rise to each case (none, unique, or many solutions).

Remark: We emphasize that, in our context, whenever we talk of a 'solution' we mean a solution in the field of real numbers, not in the field of complex numbers. But this is not at all a restriction. In fact, as long as A and b are real, then any complex solution to $Ax = b$ has to be real. Why so? Simply because if x was nonreal, while A real, then Ax would be nonreal, contradicting $Ax = b$ and b real.

Example 47 *Consider the following three* 2×2 *linear systems:*

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 1 \end{aligned} \tag{3}$$

or

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 &= 1 \end{aligned} \tag{4}$$

or

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \tag{5}$$

The question we ask is: Why does (3) admit no solution at all, (4) only one solution, and (5) a continuum of solutions? [Can you verify that claim? Can you find the set of solutions yourself?]

3.2 Nonlinear Equation Systems

Let f_i be a real function with domain $\mathbb{R}^n$, or a subset of it, let $b_i \in \mathbb{R}$, for $i = 1, ..., m$; also let $x = (x_1, ..., x_n)$ in the intersection of the domains of all f_i's. Then

$$f_i(x_1, ..., x_n) = b_i \quad \text{or} \quad f_i(x) = b_i$$

is an equation in x for each $i = 1, ..., m$. The set of m such equations,

$$\begin{aligned} f_1(x) &= b_1 \\ &\cdots \\ f_m(x) &= b_m \end{aligned} \tag{6}$$

forms a general, possibly nonlinear, system of m equations in n unknowns.

If we let b be the $m \times 1$ column vector of b_i's, and F be the vector-valued function (with values in $\mathbb{R}^m$) defined by

$$F(x) \equiv \begin{bmatrix} f_1(x_1, ..., x_n) \\ \vdots \\ f_m(x_1, ..., x_n) \end{bmatrix}$$

then we can write (6) as

$$F(x) = b$$

The set of solutions is then

$$X^* \equiv X^*(F, b) \equiv \{x \in \mathbb{R}^n | F(x) = b\}$$

In general X^* may be empty (no solution), or be a singleton (unique solution), or have more than one elements (multiple solutions).

Notice that a linear system is just the special case where F is a **linear transformation**, meaning $F(x) = Ax$ for some matrix A.

Example 48 *(Amemiya 1985) Empirical work often involves estimating a system of nonlinenear simultaneous equations. Such a system (with N equations) is defined by*

$$f_{it}(\mathbf{y}_t, \mathbf{x}_t, \boldsymbol{\alpha}_i) = u_{it}, \qquad i = 1, 2, ..., N, \qquad t = 1, 2, ..., T$$

where $\mathbf{y}_t$ is an N-vector of endogenous variables, $\mathbf{x}_t$ is a vector of exogenous variables, and $\boldsymbol{\alpha}_i$ is a K_i-vector of unknown parameters to be estimated. In the base case it is assumed that the N-vector $\mathbf{u}_t = (u_{1t}, u_{2t}, ..., u_{Nt})'$ is an i.i.d. vector random variable with zero mean and variance-covariance matrix $\mathbf{\Sigma}$. Not all of the elements of vectors $\mathbf{y}_t$ and $\mathbf{x}_t$ may actually appear in the arguments of each f_{it}. We assume that each equation has its own vector of parameters $\boldsymbol{\alpha}_i$ and that there are no constraints among $\boldsymbol{\alpha}_i$'s, but the subsequent results can easily be modified if each $\boldsymbol{\alpha}_i$ can be parametrically expressed as $\boldsymbol{\alpha}_i(\boldsymbol{\theta})$, where the number of elements in $\boldsymbol{\theta}$ is less than $\sum_{i=1}^{N} K_i$. Strictly speaking, this is not a complete model by itself because there is no guarantee that a unique solution for $\mathbf{y}_t$ exists for every possible value of u_{it} unless some stringent assumptions are made on the form of f_{it}. Therefore we assume either that f_{it} satisfies such assumptions or that if there is more than one solution for $\mathbf{y}_t$, there is some additional mechanism by which a unique solution is chosen.

3.3 Solution of Linear System of Equations

We now return to linear systems. Consider the $m \times n$ **system**:

$$Ax = b$$

where $A = [a_j] = [a_{ij}]$ is the $m \times n$ matrix of coefficients, $x = [x_j]$ is the $n \times 1$ vector of unknowns, and $b = [b_i]$ the $m \times 1$ vector of constants; let also $a_j \in \mathbb{R}^m$ be the j-th column of A, so that $A = [a_1 ... a_n]$.

What does $b = Ax$ means? Notice that

$$Ax = x_1 a_1 + x_2 a_2 + ... + x_n a_n$$

This is just a linear combination of the columns a_j's of A, with the x_j's being the corresponding weights. Therefore, $b = Ax$ simply means that the given vector $b \in \mathbb{R}^m$ can be written as a linear combination of the columns a_j's of A. Equivalent, $b = Ax$ means that b falls into the subspace spanned by A.

But recall that b falls into the span of A, and can be written as a linear combination of the a_j's, if and only if the matrix formed by stacking b together with all a_j's is singular, which also means that its span coincides with that of A alone.

Thus, letting

$$\begin{aligned} S(A) &\equiv S[a_1, ..., a_n] \equiv \\ &\equiv \left\{ y \in \mathbb{R}^m \mid \ y = Ax = \textstyle\sum_{j=1}^n x_j a_j \ \ \text{for some}\, x = [x_j] \in \mathbb{R}^n \right\} \end{aligned}$$

be the span of $A = [a_j]$ and

$$[A, b] = \begin{bmatrix} a_{11} & ... & a_{1n} & b_1 \\ a_{21} & ... & a_{2n} & b_2 \\ ... & & ... & ... \\ a_{m1} & ... & a_{mn} & b_m \end{bmatrix}$$

be the **bordered matrix** of coefficients, and by appealing to Theorem 44, we have:

Lemma 49 *The set of solutions to $Ax = b$ is nonempty if and only if b falls into the span of A; and this holds if and only if the bordered matrix $[A, b]$ is of the same rank with A, or equivalently spans the same space with A:*

$$\begin{aligned} X^*(A, b) \neq \emptyset \quad &\Leftrightarrow \quad b \in S(A) \\ &\Leftrightarrow \quad S[A, b] = S(A) \\ &\Leftrightarrow \quad rank[A, b] = rank(A) \end{aligned}$$

Also, recall that, given an $m \times n$ matrix A, the span $S(A)$ and the nullspace $N(A')$ of A form an orthogonal partition for the whole $\mathbb{R}^m$. Thus the set of solutions is empty if and only if the residual of the projection of b on $S(A)$ is nonzero, or equivalently b is not orthogonal to $N(A')$.

We now consider two complementary subclasses of linear systems: those that have $b = 0$, and those that have $b \neq 0$. We call the former **homogenous** and the latter **non-homogeneous**.

3.4 Homogeneous Linear Systems

We here consider linear systems $Ax = b$ with $b = 0$. Such systems are called **homogeneous**.

Recall that $Ax = b$ has a solution (at least one) if and only if $b \in S(A)$; here, if and only if $0 \in S(A)$. Observe then that, since the span $S(A)$ of any matrix A is a subspace, the zero vector is always an element of the span of $S(A)$. Thus, if $b = 0$, the system $Ax = 0$ always has a solution, whatever the matrix A is. Indeed, $x = 0$ is always a solution to $Ax = 0$:

$$b = 0 \Rightarrow X^* \ni 0 \tag{7}$$

Moreover, by definition indeed, for a homogeneous system we have that

$$b = 0 \Rightarrow X^* = \{x|Ax = 0\} \equiv N(A)$$

so that the set of solutions to $Ax = 0$ is simply the nullspace of the coefficient matrix A. And since $N(A)$ is a vector space as well, $0 \in N(A)$, which is an alternative way to say (7).

But in general $x = 0$ may not be the unique solution. When will $x = 0$ be the unique solution to $Ax = 0$? From Theorem 44 we know that $Ax = 0$ implies that $x = 0$ necessarily, if and only if all the columns a_j of A are linearly independent. That is, $x = 0$ is the unique solution to $Ax = 0$ if and only if $rank(A) = n$. In this case, the span of A is the whole space, $S(A) = \mathbb{R}^n$ and $rank(A) = \dim(\mathbb{R}^n)$, and its nullspace is zero-dimensional, $N(A) = \{0\}$ and $null(A) = 0$.

But we know that $rank(A) \leq \min\{n, m\}$, and thus $rank(A) = n$ implies $m \geq n$ necessarily. That is, for the solution to be unique we need at least as many equations as unknowns.

It follows that if there are **less equations than unknowns**, $m < n$, then $Ax = 0$ must have more than one solutions. With $m < n$, indeed, the columns of A are necessarily linearly dependent; in particular, the nullspace of A has dimension at least 1, and exactly as many as $null(A) = n - rank(A) \geq 1$ columns of A can be written as linear combinations of the rest.

In this case, $Ax = 0$ has a nonzero solution as well. Since $Ax = 0$ implies $Az = 0$ for any $z \in S(x)$, and $x \neq 0 \Rightarrow S(x) = \mathbb{R}$, it follows that $Ax = 0$ has indeed a continuum of solutions. More precisely, if we can find as many as k linearly independent solutions $x^1, ..., x^k$ to $Ax = 0$, then any $z \in S[x^1, ..., x^k]$ is a solution as well:

Exercise 50 *Prove the above claim. That is, prove that* $Ax^1 = Ax^2 = 0 \Rightarrow Az = 0 \, \forall z \in S[x^1, x^2]$

We observe that the maximum number k of independent solutions $x^1, ..., x^k$ to $Ax = 0$ is simply (by definition indeed) the dimension of the nullspace $N(A) \equiv \{x|Ax = 0\}$ of A; that is $k = null(A) = \dim(N(A))$ of A. Recall then that, for an $m \times n$ matrix A, it is true that $null(A) = n - rank(A)$.

Therefore, there are as many independent solutions to $Ax = 0$ as $k = n - rank(A)$. That is:

Lemma 51 *For any homogeneous system $Ax = 0$,*

$$X^* = N(A) \quad and \quad \dim(X^*) = n - rank(A)$$

Further, if $m > rank(A)$, then as many as $m - rank(A)$ equations are redundant, in the sense that they are implied by the rest $rank(A)$ equations. Thus, any $m \times n$ homogeneous system with $m > \rho$ can be reduced to an $\rho \times n$ system, where $\rho = rank(A)$.

Therefore, without any loss of generality, from now on consider only $m \times n$ systems with $m = \rho = rank(A) \leq n$. We can then distinguish two cases: either $m = n = rank(A)$, or $m = rank(A) < n$. Then $m = rank(A)$ is the 'effective' number of equations (that is, the number of linearly independent equations), while n is the number of unknowns.

- **Case I:** $n = m = rank(A)$
  In this case we have as many equations as unknowns and A is a nonsingular $n \times n$ matrix. Then, $Ax = 0$ if and only if $x = 0$. Hence, $x = 0$ is the unique solution to $Ax = 0$, $X^* = \{0\}$, and $\dim(X^*) = 0 = n - m$.

- **Case II:** $n > m = rank(A)$
  In this case we have more unknowns than equations and A is a singular $m \times n$ matrix with $rank(A) = m < n$. Then, $Ax = 0$ has as many independent solutions as $k = n - rank(A) = n - m$. This means that we may freely choose $n - m$ values for, say, the first $n - m$ unknowns $(x_1, ..., x_{n-m})$ and then the system $Ax = 0$ pins down the values for the rest m unknowns $(x_{n-m+1}, ..., x_n)$. And then $\dim(X^*) = n - m \geq 1$.

So, now let us generalize to arbitrary number of equations and unknowns. Let A be an $m \times n$ matrix for arbitrary m, n and let $\rho \leq \min\{m, n\}$ be its rank. Then, partition A and x as follows:

$$A = \begin{bmatrix} D & B \\ C & \tilde{A} \end{bmatrix} \qquad x = \begin{bmatrix} z \\ \tilde{x} \end{bmatrix} \tag{8}$$

where $\tilde{A}$ is a full-rank $\rho \times \rho$ matrix, for $\rho = rank(A) = rank(\tilde{A})$, $z = (x_1, ..., x_{n-\rho})$ is $(n-\rho) \times 1$ and $\tilde{x} = (x_{n-\rho+1}, ..., x_n)$ is $\rho \times 1$. Check the dimensions of B, C, D, and notice that

$$Ax = \begin{bmatrix} D\,z + B\,\tilde{x} \\ C\,z + \tilde{A}\,\tilde{x} \end{bmatrix}$$

so that

$$Ax = 0 \Leftrightarrow \left\{ \begin{array}{c} D\,z + B\,\tilde{x} = 0 \\ C\,z + \tilde{A}\,\tilde{x} = 0 \end{array} \right\}$$

As we explained before, the first $m - \rho$ equations are redundant. That is, $Cz + \tilde{A}\tilde{x} = 0$ implies $Dz + B\tilde{x} = 0$ as well. Thus,

$$Ax = 0 \Leftrightarrow Cz + \tilde{A}\,\tilde{x} = 0$$

Since $\tilde{A}$ has full rank, it is invertible, and therefore we get

$$Ax = 0 \Leftrightarrow \tilde{x} = -\tilde{A}^{-1}Cz$$

This means that any $x = (z, \tilde{x})$ such that $\tilde{x} = -\tilde{A}^{-1}Cz$, for any $z \in \mathbb{R}^{n-\rho}$, is a solution to $Ax = 0$. Therefrom it also follows that $\dim(X^*) = n - \rho$. And conversely, x is a solution to $Ax = 0$ only if a partition like the above is possible.

Therefore, we can summarize our results so far in the following theorem:

Theorem 52 *Consider the $m \times n$ **homogeneous** system $Ax = 0$. The **set of solutions** always includes 0 and thus is nonempty; and is given by the nullspace of A:*

$$X^* = N(A) \equiv \{x|Ax = 0\}$$

The dimension of X^* is simply the nullity of A:

$$\dim(X^*) = null(A) = n - rank(A) \geq 0$$

Whenever $m > rank(A)$, as many equations as $m - rank(A)$ are redundant. Further, the solution is unique at $x = 0$ if and only if A is of full rank,

$$X^* = \{0\} \Leftrightarrow rank(A) = n$$

Otherwise, there is a continuum of solutions of the form

$$X^* = \left\{x \in \mathbb{R}^n \,|\, x = (z, -\tilde{A}^{-1}Cz) \text{ for some } z \in \mathbb{R}^{n-rank(A)}\right\}$$

with $\tilde{A}$ being any square submatrix of A with $rank(\tilde{A}) = rank(A)$ and C then being as in (8).

Exercise 53 *Consider the 3×3 system $Ax = 0$ for*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Show that $rank(A) = 2$, and partition A appropriately so as to apply what we did before. What is the set of solutions?

3.5 Non-homogeneous Linear Systems

In the previous subsection we consider linear equation systems with $b = 0$. Now consider systems

$$Ax = b \quad \text{for} \quad b \neq 0$$

We repeat that existence of a solution means that $b \neq 0$ can be written as a linear combination of the columns in A, or that b falls into the span of A.

Consider first the case that $m = n = rank(A)$. Then A is square and has full rank, so that it is nonsingular and is a basis for the whole $\mathbb{R}^n$. It follows that $b \in S(A)$ necessarily. Moreover, since A is invertible,

$$Ax = b \;\Leftrightarrow\; x = A^{-1}b$$

Thus in this case the set of solutions is singleton, $X^* = \{A^{-1}b\}$. The result works conversely as well, and even if $b = 0$. Thus we have

Lemma 54 *A square $n \times n$ system $Ax = b$ has a unique solution if and only if A has full rank, which means that A is nonsingular, or equivalently $|A| \neq 0$. Then, $x = A^{-1}b$ is the unique solution.*

Remark: Notice that, in the above case, A is a basis for $\mathbb{R}^n$, where b belongs. Hence the geometric interpretation of the solution is that $x = A^{-1}b$ gives the (unique) coordinates of b with respect to the basis A.

Now suppose that A is not of full rank, but still $rank[A, b] = rank(A)$. This still implies $b \in S(A)$, and at least one solution exists. But now (with $b \neq 0$) the solution is not unique. Instead, we have a whole continuum of solutions!

On the other hand, letting $[A, b]$ be the bordered matrix formed as

$$[A, b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

we observe that if $S[A, B]$ is strictly bigger than $S(A)$, which is equivalent to $rank[A, b] > rank(A)$, then it must be the case that b can not be written as a linear combination of the columns of A; that is, $b \notin S(A)$ and rather the projection of b on $N(A)$ is nonzero. Therefore:

Lemma 55 *The (nonhomogeneous) system $Ax = b$ has no solution if and only if the rank of $[A, b]$ exceeds that of A,*

$$X^* = \emptyset \Leftrightarrow rank[A, b] > rank(A)$$

The situation is indeed similar to the homogeneous case. In particular, we may rewrite $Ax = b$ equivalently as

$$[\, A \,,\, b \,]\; y \;=\; 0$$

where $y = \left[\begin{smallmatrix} x \\ -1 \end{smallmatrix}\right] = (x_1, ..., x_n, -1)$. Notice that $[A, b]$ is $m \times (n + 1)$ and y is $(n+1) \times 1$, with $y \neq 0$ by construction.. This way we have in fact transformed the non-homogeneous system $Ax = b$ to a homogeneous one, $[A, b]y = 0$. The important constraint is only that we require, by construction indeed, that $y \neq 0$. Thus, for $Ax = b$ to have any solution we need that $[A, b]y$ has a non-zero solution. But the latter, as we showed before, is possible if and only if the bordered matrix $[A, b]$ is singular. If instead $[A, b]$ is nonsingular, and $null[A, b] = 0$, then $Ax = b$ has no solution.

Moreover, if $null[A, b] = 1$, then the set of solutions y of $[A, b]y = 0$ is a single-dimensional line, and thus $Ax = b$ has a unique solution. In particular, the point y of this line that has -1 as its last coordinate gives us the unique solution to $Ax = b$. In fact:

Exercise 56 *Show that $null[A, b] = 1$ if and only if $rank[A, b] = rank(A)$.*

If $null[A, b] \geq 2$, then the set of solutions y of $[A, b]y = 0$ is a hyperplane of dimension equal to $null[A, b] \geq 2$, and thus the set of solutions x of $Ax = b$ is a hyperplane with dimension equal to $null[A, b] - 1 \geq 1$.

Exercise 57 *Persuade yourself that, if $[A, b]$ is singular, then and only then $X^* \neq \emptyset$, and further $\dim(X^*) = null[A, b] - 1$.*

We can thus summarize our results in the following theorem:

Theorem 58 *Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:*

- ***(Unique Solution)*** *If $rank[A, b] = rank(A) = n \leq m$, then and only then the system has a unique solution. In this case, indeed, as many as $m - n$ equations are redundant, and, provided an appropriate partition, $X^* = \{\tilde{A}^{-1}\tilde{b}\}$.*
- ***(No Solution)*** *If $rank[A, b] > rank(A)$, which necessarily implies $b \neq 0$ and $m > rank(A)$, then and only then the system has no solution, $X^* = \emptyset$.*
- ***(Multiple Solutions)*** *If $rank[A, b] = rank(A)$ but $rank(A) < n$, then and only then the system has multiple solutions, and then $\dim(X^*) = null[A, b] - 1 = n - rank(A) \geq 1$.*

When there is a unique solution, we say that the system is **exactly determined**. When there is no, the system in **overdetermined**. When there are many solutions, the system is **underdetermined** (or indeterminate).

Exercise 59 *Let $m = n$, and consider $Ax = b$. Suppose that $|A| = 0$, so that the system is either underdetermined or overdetermined. What of the two cases arises if $b = 0$? And what happens if $b \neq 0$? Next consider $m > n = rank(A)$ and characterize the appropriate partition that gives $X^* = \{\tilde{A}^{-1}\tilde{b}\}$.*

3.6 Finding the Solution: Cramer's Rule

We have identified the conditions under which a square system $Ax = b$ has a unique solution: This is so if and only if A is invertible. Then and only then the unique solution is given by

$$x = A^{-1}b$$

Calculating this requires that we first calculate the inverse A^{-1}. This can be done with the algorithm that we presented in Subsection 2.9; the inverse of A is then given as

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} adjA = \\ &= \frac{1}{|A|} \begin{bmatrix} +|A_{11}| & -|A_{21}| & ... & (-1)^{n+1}|A_{n1}| \\ -|A_{12}| & +|A_{22}| & ... & (-1)^{n+2}|A_{n2}| \\ ... & ... & & ... \\ (-1)^{n+1}|A_{1n}| & (-1)^{n+2}|A_{1n}| & ... & +|A_{nn}| \end{bmatrix} \end{aligned}$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix formed by erasing the i-th raw and the j-th column of A, and $|A_{ij}|$ is the (i,j) minor of A.

An alternative way to calculate the solution is to use the Cramer Rule. Let B_j be the $n \times n$ matrix formed by taking A and substituting its j-th column, a_j, with the constants vector, b. For instance, for $j = 2$,

$$B_2 = [a_1 \, b \, a_3 ... a_n] = \begin{bmatrix} a_{11} & b_1 & a_{13} & ... & a_{1n} \\ a_{21} & b_2 & a_{23} & ... & a_{2n} \\ ... & ... & ... & & ... \\ a_{n1} & b_n & a_{n3} & ... & a_{nn} \end{bmatrix}$$

and so on. Let $|A| \neq 0$ and $|B_j|$ be the determinants of A and B_j, respectively. Cramer's rule then says that the j-th element of the solution $x = A^{-1}b$ is given by

$$x_j = \frac{|B_j|}{|A|} \;\; \forall j = 1, ..., n$$

♠ *Cramer's rule is good to know, but if you ever have to invert a numerical matrix, you'd better turn to Matlab/Mathematica.*