14.102, Math for Economists
Fall 2004
Lecture Notes, 9/28/2004

These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples. The usual disclaimer applies; questions and comments are welcome.

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6 Calculus: Some Preliminaries

Before we can begin our study of functions and calculus (with the ultimate goal being the study of optimization), we must bring in a few concepts from real analysis. For the moment we will state most of them without proof, just so we can use them as tools for understanding calculus; later in the course we will focus on proving these and other statements. I should also note that some of what you will see here is less general than what you would find in a math course not specifically geared towards economists - but I will try to note when this is the case.

6.1 Sequences and Limits

The concept of a sequence is very intuitive - just an infinite ordered array of real numbers (or, more generally, points in $\mathbb{R}^n$) - but is defined in a way that (at least to me) conceals this intuition.

One point to make here is that a sequence in mathematics is something *infinite*. In our everyday language, instead, we sometimes use the word "sequence" to describe something finite (like "sequence of events", for example).

Definition 106 *A (finite) number A is called the limit of sequence $\{a_n\}$ if $\forall \varepsilon > 0 \;\; \exists N : \forall n > N \;\; |a_n - A| < \varepsilon$. If such number A exists, the sequence is said to be convergent.*

Verbally, A is the limit of $\{a_n\}$ if the sequence comes closer and closer to A as N grows and, moreover, *stays* close to A "forever". Of course, such A does not have to exist, as the following simple example shows:

Example 107 *Let $a_n = (-1)^n$. Then $\{a_n\}$ does not have any limit.*

That is, a sequence does not have to converge to any single point (for example, it can oscillate between two different points). However, what it surely can never do is to converge to two distinct points at a time:

Lemma 108 *A sequence can have at most one limit.*

Definition 109 *Sequence $\{a_n\}$ is said to converge to ∞ (with no sign) if $\forall C$ $\exists N : \forall n > N \quad |a_n| > C$.*

To be convergent is a strong condition on $\{a_n\}$; to have a limit point is a weaker condition. The price you have to pay for relaxing this (or any) condition is that now more points will fit - for example, a sequence can have only one limit (which adds some desired definitiveness to the concept) but multiple limit points. What you hope to get in return is that more sequences have limit points than have limits[7]. To make an exact statement we need one more

Definition 110 *Sequence $\{a_n\}$ is called bounded if $\exists C : \forall n \quad |a_n| < C$*

6.2 Metrics and Norms

Whenever we are talking about a set of objects in mathematics, it is very common that we have a feeling about whether two particular objects are "close" to each other. What we mean is usually that the *distance* between them is small. Although it may be intuitive what the distance between two points is, it is not always that intuitive in a more general setup: for instance, how would you think about the distance between two continuous functions on the unit interval? Between two optimal control problems? Between two economies? Between two preference relations? Here is how we formalize what a distance means:

Definition 111 *A metric on $\mathbb{R}^n$ is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\forall x, y, z \in \mathbb{R}^n$:*

- *$d(x, y) \geq 0$ (we do not want negative distance),*
- *$d(x, y) = 0 \iff x = y$ (moreover, we want strictly positive distance between distinct points),*
- *$d(x, y) = d(y, x)$ (symmetry),*
- *$d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).*

Example 112 $d_1(x, y) = |x_1 - y_1| + ... + |x_n - y_n|$

Example 113 $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$ *(this is called Euclidean distance - and that is the default in $\mathbb{R}^n$)*

[7] A similar tradeoff arises in game theory: we can use strictly dominant strategies or Nash equilibrium as a solution concept; the former is more definite and probably more appealing, but need not (and in most interesting cases does not) exist; the latter always exists (for finite games at least) but need not be unique and deserves further justification. Now that, after a number of years in economics, I have finally learned the fundamental concept of tradeoff, I am amazed to see in how many instances it is applicable in math.

6.3 Open and Closed Sets

For the rest of the analysis we stick to the Euclidean metric on $\mathbb{R}^n$: $d(x,y) = d_2(x,y)$.

Definition 114 *For any $x_0 \in \mathbb{R}^n$ and $r > 0$ define an open ball $B_r(x_0) = \{x \in \mathbb{R}^n | d(x,x_0) < r\}$.*

Exercise 115 *What do open balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like? What would they look like if we fixed another metric (d_1 or d_∞) instead of d_2?*

Definition 116 *Set $A \subset \mathbb{R}^n$ is called open if, together with any point $x_0 \in A$, it contains a small enough open ball $B_\varepsilon(x_0)$ for some $\varepsilon > 0$.*

Example 117 *An open ball is an open set (why?)*

Example 118 *The half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ is open*

Definition 119 *A set is called closed if its complement is open.*

Example 120 *A closed ball $B_r(x_0) = \{x \in \mathbb{R}^n | d(x,x_0) \leq r\}$ is a closed set.*

Exercise 121 *Show that empty set $\emptyset$ and the entire space $\mathbb{R}^n$ are both open and closed. Persuade yourself that these two are the only sets which are both open and closed.*

Definition 122 *A set in $\mathbb{R}^n$ is called compact if it is closed and bounded.*

This is not the traditional definition of compactness that you will find in a textbook – in spaces more general than $\mathbb{R}^n$ it will not work (that is, in those spaces there exist closed and bounded sets which will not be compact). However, in $\mathbb{R}^n$ it will work fine: whatever definition of compactness you will ever see, it will be equivalent to the one above.