

14.102, Math for Economists
Fall 2004
Lecture Notes, 9/30/2004

These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples. The usual disclaimer applies; questions and comments are welcome.

Nathan Barczy
nab@mit.edu

7 Functions and Calculus

Typically mathematicians rely on the notion of a *function* as being built in to general perception of abstract mathematical objects, which means that they do not bother to rigorously define it. However, given Kantor's program of redefining everything in terms of sets only, functions also needed to be so defined. And the way Kantor did it was simple: he just identified a function with its graph. Namely, a *function* (for example, from \mathbb{R}^n to \mathbb{R}^m) is a subset $graph(f)$ of the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ with one restriction that any $x \in \mathbb{R}^n$ may be mapped only to one $y \in \mathbb{R}^m$, i.e., that $(x, y_1) \in F \& (x, y_2) \in F \implies y_1 = y_2$. This last property allows us to define $y = y(x)$ unambiguously.

Note that according to the definition above a function does not have to map the entire space \mathbb{R}^n to \mathbb{R}^m , i.e., it does not have to be the case that $\forall x \in \mathbb{R}^n \exists y \in \mathbb{R}^m$ such that $(x, y) \in graph(f)$. If such y does exist for a given x (in which case it is unique by definition), we say that x belongs to the *domain* of f .

Exercise 123 Find the domains of the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$:

1. $f(x) = \sqrt{x}$
2. $f(x) = \frac{1}{x^2 + 2x - 3}$
3. $f(x) = \frac{1}{\sin x} + \frac{1}{\cos x}$

A subset of $\mathbb{R}^n \times \mathbb{R}^m$ can easily be converted to a subset of $\mathbb{R}^m \times \mathbb{R}^n$ by switching coordinates. The resulting subset does not have to be a function (in the sense that there does not have to be unique x associated with each y), but if it is a function it is called the *inverse* function and is denoted $f^{-1}(y)$.

The above definitions are quite abstract, but the main idea of a function is probably clear to everybody.

The concept of a function (usually in terms of one variable being uniquely determined by another, in our case y being determined by x) has been understood for a very long time, by the ancient Greeks at least (although they did not quite operate in terms of *variables*, let alone using letters to denote variables, which was introduced by a French mathematician Vieta in late XVI century for purposes entirely different from those we use them for nowadays). The birth of modern calculus is dated back to Newton and Leibniz I think (in the mid XVII century) and the fundamental notion they managed to give a formal definition for was *continuity*. Informally, a continuous function is a function, whose graph we can draw in one touch of a pencil, without taking it off the paper. Although intuitively quite appealing, this “definition” is not operative, so we need a formal (and, of course much less intuitive) one:

Definition 124 A function $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be continuous at point $x_0 \in X$ if $\forall \varepsilon > 0 \exists \delta > 0$ such that as $f(x) \in B_\varepsilon(f(x_0)) \subset \mathbb{R}^m \forall x \in B_\delta(x_0)$. A function is said to be continuous on X if it is continuous at each point of X .

Example 125 $f(x) = \text{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ is continuous at all points except 0.

Example 126 $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not continuous anywhere (it is called Dirichlet function after another French mathematician Lejeune Dirichlet, 1805-1859)

Example 127 $g(x) = xD(x)$, where $D(x)$ is Dirichlet function defined above, is continuous at $x_0 = 0$ and nowhere else.

Example 128 $R(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ with minimum possible } p \text{ and } q \\ 0, & x \notin \mathbb{Q} \end{cases}$ is continuous at all irrational points $x \notin \mathbb{Q}$ and not continuous at rational points (it is called Riemann function after a German mathematician Georg Riemann, 1826-1866).

Exercise 129 Make sure that you fully understand the examples above.

Example 130 Usual functions are continuous on their domains: \ln , \exp , \sin , \cos , polynomials, radicals.

Lemma 131 A sum, a difference and a product of two continuous functions are continuous; a ratio of two continuous functions is continuous at all points where the denominator does not vanish.

Let f be a strictly increasing continuous function on $[a, b]$ then $f([a, b]) = [f(a), f(b)]$ and f is a one-to-one function from $[a, b]$ onto $[f(a), f(b)]$.

Then the inverse f^{-1} is a one-to-one function from $[f(a), f(b)]$ onto $[a, b]$.

Theorem 132 *If f is continuous and increasing then so is f^{-1} .*

Draw graph and note that f and f^{-1} are symmetric with respect to the 45° line.

Definition 133 *A function on $I = [a, b]$ bounded interval is **piece-wise continuous** if it is continuous everywhere except on a finite number of points in I and that at every point where it is not continuous it admits left and right limits.*

7.1 First Optimization Result: Weierstraß Theorem

Theorem 134 (*Weierstraß*) *If $X \subset \mathbb{R}^n$ is a compact set and $f : X \rightarrow \mathbb{R}$ is a continuous function, then f attains a maximum on X , that is, there exists a point $x^* \in X$ such that $\forall x \in X \quad f(x) \leq f(x^*)$.*

Corollary 135 *... and it attains a minimum.*

Note: this result is guarantees an existence of an optimal solution to many economic problems.

We may generalize it to compact subsets of R^n . A set $A \subseteq R^n$ is called compact if it is closed and bounded. A set is closed if it contains all limits of sequences of its points.

This result is the cornerstone of all optimization theory. It guarantees that there definitely exists an optimum in a wide class of problems that we will see when we study economics. Besides, it reveals why we prefer dealing with compact (and hence closed) sets, despite them being more "difficult" than open sets (as closed sets typical have points of two kinds – interior and boundary – while open sets only have points of the former kind): on open sets optimization is not guaranteed to be successful, as the following examples show.

Example 136 *Let $X = (0, 1)$ and $f(x) = \frac{1}{x}$. The set is bounded and the function is continuous on it, but it attains no maximum. That is because the set is not closed.*

Example 137 *Let $X = [0, +\infty)$ and $f(x) = x^2$. The set is closed and the function is continuous on it, but it attains no maximum. That is because the set is not bounded.*

Example 138 *Let $X = [0, 1]$ and $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$. Then the set is closed and bounded, but the function attains no maximum. That is because it is not continuous.*

Example 139 *Let $X = (0, +\infty)$ and $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. Then the set is neither closed nor bounded, and the function is not continuous, and yet it attains a maximum. That is because conditions stated in the theorem are sufficient but not necessary.*

Exercise 140 Consider standard consumer optimization problem:

$$\begin{aligned} & \max u(x_1, x_2) \\ \text{s.t. } & p_1x_1 + p_2x_2 = I \end{aligned}$$

What restrictions on parameters p_1, p_2 and I are sufficient for Weierstraß theorem to be applicable? What about on x_1 and x_2 ?

Theorem 141 Intermediate Value Theorem Let f be a continuous function on a closed interval $[a, b]$. Then for any m in the interval $[f(a), f(b)]$ or $([f(b), f(a)])$, there is some $c \in [a, b]$ such that $f(c) = m$.

Proof. Suppose $f(a) < f(b)$. Let $S = \{x \in [a, b] | f(x) \leq m\}$, S non empty and bounded. So consider $c = \sup S$. Take x_n converging to c , of course $f(x_n) \leq m$ so $f(c) \leq m$. Now take $x_n > c$ converging to c . $f(x_n) > m$ so the limit $f(c) \geq m$. So $f(c) = m$. ■

Example: existence of an equilibrium price. If $Supply(0) = Demand(+\infty) = 0$ and $Supply(+\infty) = Demand(0) = +\infty$ then $\exists p / Supply(p) = Demand(p)$.

This example is related to fixed point theorems:

Theorem 142 (Brouwer's) Let A be a convex and compact subset of R (or R^n) and let $f : A \rightarrow A$ be a continuous function. Then, there exists a fixed point of f that is a point $x \in A$ such that

$$f(x) = x$$

A set $A \subseteq R^n$ is called convex if $a, b \in A$ implies that the entire segment between a and b is contained in A .

Theorem 143 (Tarski's) Let A be a set with ordering \leq such that each subset of A has a sup and an inf. Let $f : A \rightarrow A$ be monotonic (weakly increasing or decreasing). Then, f has a fixed point.

Theorem 144 (Banach's) Let A be a space with metrics $d(\cdot, \cdot)$ and $f : A \rightarrow A$ be a contraction that is there exists $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Then, f has a unique fixed point.

A (Kakutani's) version of the Brouwer theorem is used to prove the existence of Nash equilibria in many games. Tarski's theorem is used to prove existence of equilibria of supermodular games. Banach's theorem is the workhorse of the theory of differential equations that will be studied in 14.102.

7.2 Convexity

Definition 145 $f : I \rightarrow R$ a function on I . f is convex iff for any x and y in I and any $0 < \lambda < 1$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

f is concave iff for any x and y in I and any $0 < \lambda < 1$

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

Useful facts:

- f concave iff $-f$ convex (idem for strictly)
- sum of convex functions is convex.
- sum of concave functions is concave.

Convex (concave) function is below (above) the segment $[(x, f(x)), (y, f(y))]$.

Convexity guarantees uniqueness of a maximum (provided a maximum exists). Convexity of a continuous function on a compact domain in R^n guarantees (a) existence of the maximum and (b) uniqueness of the maximum. This is often used in maximization problems.

What about concave functions?