14.102, Math for Economists
Fall 2004
Lecture Notes, 10/14/2004

These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples. The usual disclaimer applies; questions and comments are welcome.

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12 Basic Integration in R

A function on a bounded interval $[a, b]$ is piece-wise continuous if it is continuous everywhere except on a finite number of points in I and that at every point where it is not continuous it admits finite left and right limits.

Definition 197 *For piecewise continuous functions on interval I, we define for any a and b in I $(a < b)$:*

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + \frac{k}{n}(b-a))$$

By definition: $\int_b^a f(x)dx = -\int_a^b f(x)dx$

The geometric interpretation of this integral (the Riemann integral) is the area under the curve. Look at the notation. Can show that such a limit exists. There are problems with this notion of the integral; too often it is not well defined.

It is possible to extend the definition of Riemann integral to a broader class of functions. One such useful generalization is Lebesgue integral based on measure theory. First note that measure theory generalizes the notions of length, area, and volume; you will need a dose of it while studying probability theory for statistics and econometrics. Dividing $[a, b]$ into any measurable sets and using measures of set instead of lengths of intervals we can reproduce the above definition and basically construct the Lebesgue integral. The payoff from this more elaborate construction is that the limit "often" exist, and hence any "well-behaved" function is measurable and admits an integral (you may encounter some measureability problems in the stochastic dynamic programming though).

Note, an important property of the Lebesgue integral is that the integral over sets of measure 0 has value 0 which is not necessarily the case for the Riemann construction! Otherwise Lebegue's has pretty much the same properties as

Riemann's (and it gives the same number as Riemann's whenever Riemann's exists).

A key example — probability density $f(x)$ and cumulative distribution $F(x)$.

Proposition 198 *Useful properties of the integral:*

- *if* $f \geq 0$ *and* $b \geq a$, *then* $\int_a^b f \geq 0$
- $\int_a^b (f+g) = \int_a^b f + \int_a^b g$
- $\int \lambda f(x)dx = \lambda \int f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- *if* $f(x) \leq g(x)$ *for all* $x \in [a,b]$ *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
- $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$
- $|\int_a^b f(x)dx| \leq \sup|f(x)| \times |b-a|$
- $|\int_a^b f(x)g(x)dx| \leq \sup|f(x)| \int_a^b |g(x)|dx$

Useful primitives:

- $\int_a^b (1/x)dx = \ln(b) - \ln(a)$
- $\int_a^b \exp(x)dx = \exp(b) - \exp(a)$
- $\int_a^b t^\alpha dt = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}$

12.1 Fundamental Theorem of Calculus

(a) Let f be integrable on an interval I, and let $a \in I$. Let $F(x) = \int_a^x f(t)dt$. If f is continuous at x then F is differentiable at x and

$$F'(x) \equiv \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

(b) Suppose that F is differentiable on an interval I and that $F' = f$ is integrable. Then

$$\int_a^b f(x)dx = F(b) - F(a) \text{ for } a, b \in I.$$

Proof.
(a)

$$F'(x) = \frac{F(x+h) - F(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

Since, f is continuous at x so for any $\varepsilon > 0$ if h is small enough then we have

$$|f(x+h) - f(x)| < \varepsilon$$

and thus, if we define x^* as $x^* = x : |f(x^*) - f(x)| = \sup_{x' \in [x, x+h]} |f(x') - f(x)|$, we have

$$\left| \frac{F(x+h) - F(x)}{h} - \frac{hf(x)}{h} \right| = \left| \frac{\int_x^{x+h} f(t)dt}{h} - \frac{hf(x)}{h} \right| \leq \left| \frac{hf(x^*)}{h} - \frac{hf(x)}{h} \right| < \frac{h\varepsilon}{h} = \varepsilon$$

as $h \to 0$ we can take $\varepsilon \to 0$ and thus

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

(b) Let's consider only the easy case when f is continuous. Denote $G(x) = \int_a^x f(t)dt$ and notice that by (a) $G' = f$. Hence, F and G have same derivative, and $(F - G)' = 0$ and thus $F - G$ is constant, denote it c. Thus,

$$\begin{aligned} F(b) - F(a) &= (G(b) + c) - (G(a) + c) \\ &= G(b) - G(a) = \int_a^b f(t)dt. \end{aligned}$$

13 Change of variables and Integration by parts

Theorem 199 *Change of variables: Let J_1 and J_2 be intervals (with more than one point):Let $f : J_1 \to J_2$ and $g : J_2 \to R$ continuous. Assume that f is differentiable and f' continuous. Then for any a, b in J_1*

$$\int_a^b g(f(x))f'(x)dx = \int_{f(a)}^{f(b)} g(u)du \tag{18}$$

Proof. Let G be a primitive of g. Then $G \circ f$ is the primitive of $x \to g(f(x))f'(x)$ by the chain rule. Both terms in (18) are equal to $G(f(b)) - G(f(a))$. ■

Example: $I = \int_0^1 2xe^{x^2}dx$, say $u = e^{x^2}$ and $du = 2xe^{x^2}$, $I = \int_1^e du = e - 1$. Be familiar with the following two important special cases:

$$\int_a^b f(t + \alpha)dt = \int_{a+\alpha}^{b+\alpha} f(u)du$$

$$\int_a^b f(\alpha t)dt = \int_{\alpha a}^{\alpha b} \frac{f(u)}{\alpha} du$$

Proposition 200 ***Integration by parts:*** *Suppose F and G are differentiable on $[a, b]$. Suppose $F' = f$ and $G' = g$. are continuous. Then:*

$$\int_a^b f(t)G(t)dt = [F(b)G(b) - F(a)G(a)] - \int_a^b F(t)g(t)dt$$

You can obtain this formula quickly by noticing that $[FG]' = fG + Fg$.

Exercise 201 *Find $\int_a^x \log(t)dt$.*

13.1 Differentiation Under the Integral Sign

Often we encounter situations under which we wish to interchange the order of integration and differentiation.

Proposition 202 ***Leibniz rule:*** *If $f(t,x)$, $a(x)$, and $b(x)$ are differentiable with respect to x, then:*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)dt = \int_{a(x)}^{b(x)} \frac{\partial f(t,x)}{\partial x}dt + b'(x)f(b(x),x) - a'(x)f(a(x),x)$$

Note that if $a(x)$ and $b(x)$ are constant, we have a special case:

$$\frac{d}{dx}\int_{a}^{b} f(t,x)dt = \int_{a}^{b} \frac{\partial f(t,x)}{\partial x}dt$$

Notice that this question really comes down to when it is justifiable to exchange the order of integration and a limit, since the derivative is a particular kind of limit. A full treatment of this question requires a bit of measure theory, which we won't go into here. However, a couple important results can be presented, all of which are variations on Lebesgue's Dominated Convergence Theorem (see Rudin; see also section 2.4 of Casella and Berger).

Theorem 203 *Suppose the function $h(x,y)$ is continuous at y_0 for each x, and there exists a function $g(x)$ satisfying*

1. *$|h(x,y)| \leq g(x)$ for all x and y,*
2. *$\int_{-\infty}^{\infty} g(x)dx < \infty$*

Then

$$\lim_{y\to y_0}\int_{-\infty}^{\infty} h(x,y)dx = \int_{-\infty}^{\infty}\lim_{y\to y_0} h(x,y)dx$$

The key condition is the existence of a dominating function $g(x)$, with a finite integral, which ensures that the integral of $h(x,y)$ cannot be too badly behaved. If we apply this to the case we are interested in, the derivative, we have

Theorem 204 *Suppose $f(t,x)$ is differentiable at $x = x_0$, that is,*

$$\lim_{h\to 0}\frac{f(t,x_0+h) - f(t,x_0)}{h} = \frac{\partial}{\partial x}f(t,x)\bigg|_{x=x_0}$$

exists for every t, and there exists a function $g(t,x_0)$, for all t and a constant $h_0 > 0$ such that

1. *$\left|\frac{f(t,x_0+h)-f(t,x_0)}{h}\right| \leq g(t,x_0)$, for all t and $|h| \leq h_0$,*

2. $\int_{-\infty}^{\infty} g(t, x_0)dx < \infty$.

Then

$$\frac{d}{dx}\int_{-\infty}^{\infty} f(t,x)dx\bigg|_{x=x_0} = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\ f(t,x)|_{x=x_0}\right]dx$$

The conditions essentially bound variability in the derivative of the function; they are similar to a smoothness condition called the *Lipschitz condition*. Most of the applications of these results which you'll see will come in statistics and econometrics, where many results in asymptotic theory examining the convergence behavior of a function as our data become infinite begin with a condition bounding the variance of the function in question.

Note that the theorem is stated for a particular value of x; often we have functions which are differentiable over some interval, and the theorem holds for x within this interval instead of a single value of x.

13.2 Improper Integrals

Remark: improper integral: if $\lim_{A\to+\infty}\int_a^A f(t)dt$ exists, we note it $\int_a^{+\infty} f(t)dt$

Example : $\int_a^{+\infty} e^{-rt}dt = [-\frac{e^{-rt}}{r}]_a^{+\infty} = \frac{e^{-ra}}{r}$

Exercise 205 *compute* $\int_a^{+\infty} te^{-rt}dt$ *(use an integration by part)*

Exercise 206 *compute* $\int_0^{+\infty} e^{-\sqrt{t}}dt$ *(use the change of variable* $u = \sqrt{t}$*)*