14.102, Math for Economists
Fall 2004
Lecture Notes, 10/19/2004

These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples, mostly from Rudin. The usual disclaimer applies; questions and comments are welcome.

Nathan Barczi
nab@mit.edu

14 Real Analysis I

We use standard notation $A \cap B$ for the intersection of two (or more) sets, $A \cup B$ for union, A^c for complement (i.e., for the set of all elements not in A).

14.1 Relations and Equivalences

For two sets A and B, whose elements can be anything whatsoever, a *relation* between two points is a function $R : A \times B \to \{0, 1\}$.We write xRy if points x and y are in relation R (i.e., $R(x, y) = 1$).

Example 207 *xRy if $x_1 > y_1$*

Example 208 *xRy if $x_1 = y_2$*

Example 209 *xRy if $\|x\| = \|y\|$*

As it turns out, there is one fundamental class of relations that is important in microeconomic theory. These are called *equivalence relations* or *equivalences.*

Definition 210 *A relation is called equivalence (usually denoted by $\sim$) if it satisfies the following three properties:*

- *$x \sim x$ (reflexive)*
- *$x \sim y \Longrightarrow y \sim x$ (symmetric)*
- *$x \sim y$ & $y \sim z \Longrightarrow x \sim z$ (transitive)*

Exercise 211 *For each of the three examples of relations above, find out whether it is reflexive, symmetric and transitive.*

Exercise 212 *Give an example of R that is symmetric but not transitive.*

Equivalence relations are essential for an axiomatic development of the utility function: for a utility function to exist, it is a necessary condition that relation "xRy if the consumer is indifferent between bundles x and y" be an equivalence (why?). Although it seems obvious that this relation is indeed an equivalence, and economic models usually assume that it is, it might not be a great description of reality. For instance, I am pretty much indifferent between my welfare now and if I give away a nickel; however, such indifference is surely not transitive: if I give away a million nickels, I will be significantly worse off.

Definition 213 *Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:*

1. *If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true.*
2. *If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.*

Note that the relation $>$ is transitive, but neither reflexive nor symmetric (indeed, it is *antisymmetric*: $x > y \Longrightarrow y \not> x$).

It is often convenient to write $x > y$ in place of $y < x$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of the two holds. In other words, $x \leq y$ is the negation of $x > y$.

14.2 Ordered Sets

Definition 214 *An **ordered set** is a set S for which an order is defined.*

Example 215 *$\mathbb{R}$ is an ordered set, but the set of all n-tuples (i.e., $\mathbb{R}^n$) is not.*

Definition 216 *Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for ever $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E.*

Lower bounds are defined in the same way, with $\geq$ in place of $\leq$.

Definition 217 *Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:*

1. *α is an upper bound of E.*
2. *If $\gamma < \alpha$ then γ is not an upper bound of E.*

*Then α is called the **least upper bound of** E or the **supremum of** E, and we write $\alpha = \sup E$.*

*The **greatest lower bound**, or **infimum**, of a set E which is bounded below is defined in the same manner: The statement $\alpha = \inf E$ means that α is a lower bound of E and that no $\beta > \alpha$ is a lower bound of E.*

Definition 218 *An ordered set S is said to have the* ***least-upper-bound property*** *if the following is true:*

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Exercise 219 *Show that $\mathbb{Q}$ does not have the least-upper-bound property.*

Theorem 220 *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.*

Proof. *Since B is bounded below, L is not empty. By the definition of L, we see that every $x \in B$ is an upper bound of L. Thus L is bounded above. Thus, $\sup L$ exists in S; call it α.*

If $\gamma < \alpha$, then γ is not an upper bound of L, so $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

Finally, note that any $\beta > \alpha$ is not in L, because α is an upper bound for L.

Thus, we have shown that $\alpha \in L$, but that any $\beta > \alpha$ is not in L. But L is the set of all lower bounds for B, so α is the greatest lower bound for B. This means precisely that $\alpha = \inf B$. ■

14.3 Finite, Countable, and Uncountable Sets(*)

Definition 221 *For any positive integer n, let J_n be the set whose elements are the integers $1, 2, ..., n$; let J be the set consisting of all positive integers. For any set A, we say:*

1. *A is* ***finite*** *if $A \sim J_n$ for some n (the empty set is also considered to be finite).*
2. *A is* ***infinite*** *if A is not finite.*
3. *A is* ***countable*** *if $A \sim J$.*
4. *A is* ***uncountable*** *if A is neither finite nor countable.*
5. *A is* ***at most countable*** *if A is finite or countable.*

Note that by convention, countable implies infinite (so, strictly speaking, we do not say 'countably infinite', although you will hear this phrase from time to time).

For two finite sets, we have $A \sim B$ iff A and B 'have the same number of elements'. But for infinite sets this notion becomes vague, while the idea of 1-1 correspondence (under which, given a mapping from A to B, the image in B of $x_1 \in A$ is distinct from the image in B of $x_2 \in A$ whenever x_1 is distinct from x_2) retains its clarity.

Example 222 *Let A be the set of all integers. Then A is countable. For consider the following arrangement of the sets A and J:*

$A : 0, 1, -1, 2, -2, 3, -3, ...$

$J : 1, 2, 3, 4, 5, 6, 7, ...$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & \text{(n even)} \\ -\frac{n-1}{2} & \text{(n odd)} \end{cases}$$

Theorem 223 *Every infinite subset of a countable set A is countable.*

Proof. *Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:*

Let n_1 be the smallest postive integer such that $x_{n_k} \in E$. Having chosen $n_1, ..., n_{k-1} (k = 2, 3, 4, ...)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Then, letting $f(k) = x_{n_k} (k = 1, 2, 3, ...)$, we obtain a 1-1 correspondence between E and J. ■

One interpretation of the theorem is that countability represents the 'smallest' kind of infinity, in that no uncountable set can be a subset of a countable set.

Theorem 224 *Let $\{E_n\}$ be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Proof. *Let every set E_n be arranged in a sequence $\{x_{nk}\}, k = 1, 2, 3, ...$ and consider the infinite array*

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & ... \\ x_{21} & x_{22} & x_{23} & ... \\ x_{31} & x_{32} & x_{33} & ... \\ ... & ... & ... & ... \end{array}$$

in which the elements of E_n form the n^{th} row. The array contains all elements of S. We can arrange these elements in a sequence as follows:

$$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}, ...$$

If any of the sets E_n have elements in common, these will appear more than once in the above sequence. Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable. Since $E_1 \subset S$, and E_1 is infinite, S is also infinite, and thus countable. ■

Theorem 225 *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, ..., a_n)$ where $a_k \in A(k = 1, 2, ..., n)$ and the elements $a_1, ..., a_n$ need not be distinct. Then B_n is countable.*
Proof. *That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable ($n = 2, 3, 4, ...$). The elements of B_n are of the form*

$$(b, a) \qquad (b \in B_{n-1}, a \in A)$$

For ever fixed b, the set of pairs (b, a) is equivalent to A, and thus countable. Thus B_n is the union of a countable set of countable sets; thus, B_n is countable, and the proof follows by induction on n.

Corollary 226 *The set of all rational numbers is countable.*
Proof. *We apply the previous theorem with $n = 2$, noting that every rational number can be written as b/a, where b and a are integers. Since the set of pairs (b, a) is countable, the set of quotients b/a, and thus the set of rational numbers, is countable.* ■

■

14.4 Metrics and Norms(*)

Whenever we are talking about a set of objects in mathematics, it is very common that we have a feeling about whether two particular objects are "close" to each other. What we mean is usually that the *distance* between them is small. Although it may be intuitive what the distance between two points is, it is not always that intuitive in a more general setup: for instance, how would you think about the distance between two continuous functions on the unit interval? Between two optimal control problems? Between two economies? Between two preference relations? Here is how we formalize what a distance means:

Definition 227 *A **metric space** is a set X, whose elements are called **points**, such that with every two points x and y belonging to X, there is a real number $d(x, y)$ associated with these two points, and called the **distance** from x to y, which satisfies:*

- *$d(x, y) \geq 0$ (we do not want negative distance),*
- *$d(x, y) = 0 \iff x = y$ (moreover, we want strictly positive distance between distinct points),*
- *$d(x, y) = d(y, x)$ (symmetry),*
- *$d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).*

Another related concept is that of a *norm.* Norm is only defined for a linear space; here we provide a definition on $\mathbb{R}^n$ (note that we never employed linear structure in our definition of metric).

Definition 228 *A norm of a vector is a function* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *such that* $\forall x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$:

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ *iff* $x = 0$,
3. $\|\lambda x\| = |\lambda| \, \|x\|$,
4. $\|x + y\| \leq \|x\| + \|y\|$.

Example 229 $\|x\| = \sqrt{x \cdot x}$, *where* $x \cdot x$ *is an inner product, is a norm (why?).*

Exercise 230 *Show that if* $\|x\|$ *is a norm, then* $d(x, y)$ *defined as* $d(x, y) = \|x - y\|$ *is a metric (it is called the metric, generated by a norm).*

Exercise 231 *For all four examples of metrics above, find out if there exists a norm that generates it.*