14.102, Math for Economists
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These notes are primarily based on those written by Andrei Bremzen for 14.102 in 2002/3, and by Marek Pycia for the MIT Math Camp in 2003/4. I have made only minor changes to the order of presentation, and added a few short examples, mostly from Rudin. The usual disclaimer applies; questions and comments are welcome.

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15 Real Analysis II

15.1 Sequences and Limits

The concept of a sequence is very intuitive - just an infinite ordered array of real numbers (or, more generally, points in $\mathbb{R}^n$) - but is defined in a way that (at least to me) conceals this intuition.

One point to make here is that a sequence in mathematics is something *infinite*. In our everyday language, instead, we sometimes use the word "sequence" to describe something finite (like "sequence of events", for example).

Definition 232 *A (finite) number A is called the limit of sequence $\{a_n\}$ if $\forall\varepsilon > 0\ \ \exists N : \forall n > N\ \ |a_n - A| < \varepsilon$. If such number A exists, the sequence is said to be convergent.*

Our next step is to capture the fact that even a divergent (i.e., nonconvergent) sequence can still have "frequently visited" or "concentration" points - points to which infinitely many terms of the sequence are "close". This intuition is captured in

Definition 233 *A (finite) number B is called a limit point of $\{a_n\}$ if $\forall\varepsilon > 0$ $\forall N : \exists n > N\ \ |a_n - B| < \varepsilon$.*

Exercise 234 *Find all limit points of the sequence in the sequence $\{a_n\}$ with $a_n = (-1)^n$. Recall that this sequence has no **limit**..*

Exercise 235 *The only limit point of a convergent sequence is its limit.*

Example 236 *The converse does not hold: consider sequence $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}$, etc. Its only limit point is 0 (why?) but it does not converge to it.*

Exercise 237 *Define the meaning of ∞ being a limit point of a sequence.*

To be convergent is a strong condition on $\{a_n\}$; to have a limit point is a weaker condition. The price you have to pay for relaxing this (or any) condition is that now more points will fit - for example, a sequence can have only one limit (which adds some desired definitiveness to the concept) but multiple limit points. What you hope to get in return is that more sequences have limit points than have limits[9]. To make an exact statement we need one more

Definition 238 *Sequence $\{a_n\}$ is called bounded if $\exists C : \forall n \ \ |a_n| < C$*

Exercise 239 *Every convergent sequence is bounded.*

Now we are ready for

Theorem 240 *(Bolzano-Weierstraß) Every bounded sequence has a limit point.*

This theorem is often stated as 'Every bounded sequence has a convergent subsequence.' The idea is simple: if a sequence has a limit point, then we know that no matter how far out into the sequence we get, we always return to an arbitrarily small neighborhood of the limit point eventually (before possibly leaving again, and returning, and leaving...). So we can construct an infinite subsequence, selecting only the points sufficiently close to the limit point, which in fact *converges* to the limit point as its *limit*.

15.2 Cauchy Sequences(*)

The 'Cauchy method' is often useful in establishing the convergence of a given sequence, without necessarily defining the limit to which it converges.

Definition 241 *A sequence $\{p_n\}$ in a metric space X is said to be a* ***Cauchy sequence*** *if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.*

Theorem 242 ***Problem 243***

1. *In any metric space X, every convergent sequence is a Cauchy sequence.*
2. *In $\mathbb{R}^k$, every Cauchy sequence converges.*

Proof.

1. If $p_n \to p$ and if $\epsilon > 0$, there is an integer N such that $d(p, p_n) < \epsilon$ for all $n \geq N$. Hence

$$d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$$

as long as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

[9] A similar tradeoff arises in game theory: we can use strictly dominant strategies or Nash equilibrium as a solution concept; the former is more definite and probably more appealing, but need not (and in most interesting cases does not) exist; the latter always exists (for finite games at least) but need not be unique and deserves further justification. Now that, after a number of years in economics, I have finally learned the fundamental concept of tradeoff, I am amazed to see in how many instances it is applicable in math.

2. (Sketch) The full proof requires concepts that we haven't had time to go into, but it can be found in Rudin, Theorem 3.11. The idea of the proof is that if $\{p_n\}$ is Cauchy, then we know that there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$. Let $n \geq N$, for the N in the hypothesis. Then $d(p_n, p_{n+1}) < \epsilon$, $d(p_{n+1}, p_{n+2}) < \epsilon$, ..., $d(p_{n+i}, p_{n+i+1}) < \epsilon$, for all $i \geq 0$. By the triangle inequality, $d(p_n, p_{n+i+1}) \leq \sum_{j=0}^{i} d(p_{n+i}, p_{n+i+1}) < n\varepsilon$. So the sequence is 'converging' to a 'limit' of p_n. Of course, the 'sketchiness' of this proof arises in the fact that p_n is not necessarily the limit of the sequence, and that we run into trouble with this argument when we let n go to infinity - but this gives about the right intuition.

■

15.3 Continuity and Upper/Lower Hemicontinuity(*)

Earlier we presented a number of fixed point theorems, including Brouwer's:

Theorem 244 *(Brouwer's) Let A be a convex and compact subset of R (or R^n) and let $f : A \rightarrow A$ be a continous function. Then, there exists a fixed point of f that is a point $x \in A$ such that*

$$f(x) = x$$

We mentioned in passing that a version of this theorem is used to prove the existence of Nash equilibria in finite games. This version, Kakutani's, weakens the conditions of Brouwer's theorem so that it applies to more games - indeed, to all finite strategic-form games. 'Finite' refers to the number of players and the actions they have to choose from; Glenn will go over this, as well as the distinction between strategic-form and extensive-form games, in more detail. He will also discuss how such games are interpreted to fit the conditions of the theorem. For now, our concern is to achieve an understanding of those conditions.

Kakutani's theorem is as follows:

Theorem 245 *(Kakutani) Let Σ be a compact, convex, nonempty subset of a finite-dimensional Euclidean space, and $r : \Sigma \rightrightarrows \Sigma$ a correspondence from Σ to Σ which satisfies the following:*

1. *$r(\sigma)$ is nonempty for all $\sigma \in \Sigma$.*

2. *$r(\sigma)$ is convex for all $\sigma \in \Sigma$.*

3. *$r(\cdot)$ has a closed graph.*

Then r has a fixed point.

Everything in this theorem is familiar from our previous discussion, with the exception of the third requirement for r, that it have a closed graph. This property is also referred to as upper-hemi continuity.

Definition 246 *A compact-valued correspondence $g : A \rightrightarrows B$ is* ***upper hemi-continuous*** *at a if $g(a)$ is nonempty and if, for every sequence $a_n \to a$ and every sequence $\{b_n\}$ such that $b_n \in g(a_n)$ for all n, there exists a convergent subsequence of $\{b_n\}$ whose limit point b is in $g(a)$.*

In words, this says that for every sequence of points in the graph of the correspondence that converges to some limit, that limit is also in the graph of the correspondence. This means that we don't 'lose points' in our graph at the limit of a convergent sequence of points in the graph, and important property for ensuring that we have a fixed point.

There is also a property called lower hemi-continuity:

Definition 247 *A correspondence $g : A \rightrightarrows B$ is said to be* ***lower hemi-continuous*** *at a if $g(a)$ is nonempty and if, for every $b \in g(a)$ and every sequence $a_n \to a$, there exists $N \geq 1$ and a sequence $\{b_n\}_{n=N}^{\infty}$ such that $b_n \to b$ and $b_n \in g(a_n)$ for all $n \geq N$.*

In words, this says that for every point in the graph of the correspondence, if there is a sequence in A converging to a point a for which $g(a)$ is nonempty, then there is also a sequence in B converging to $b \in g(a)$, and that every point b_n in that sequence is in the graph of a_n.

Together, these two give us continuity:

Definition 248 *A correspondence $g : A \rightrightarrows B$ is* ***continuous*** *at $a \in A$ if it is both u.h.c and l.h.c. at a.*

15.4 Open and Closed Sets

For the rest of the analysis we stick to the Euclidean metric on $\mathbb{R}^n$: $d(x,y) = d_2(x,y)$.

Definition 249 *For any $x_0 \in \mathbb{R}^n$ and $r > 0$ define an open ball $B_r(x_0) = \{x \in \mathbb{R}^n | d(x, x_0) < r\}$.*

Exercise 250 *What do open balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like? What would they look like if we fixed another metric (d_1 or d_∞) instead of d_2?*

Definition 251 *Set $A \subset \mathbb{R}^n$ is called* ***open*** *if, together with any point $x_0 \in A$, it contains a small enough open ball $B_\varepsilon(x_0)$ for some $\varepsilon > 0$.*

Example 252 *An open ball is an open set (why?)*

Example 253 *The half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ is open*

Exercise 254 *The union of any (not necessarily finite) number of open sets is open; the intersection of two (or any finite number of) open sets is open.*

Example 255 *Let $A_n = \{-\frac{1}{n} < x < \frac{1}{n}\}$. Persuade yourself that A_n is open for all n. What is the intersection of all $A_n, n = 1, 2, ...$? Show that it is not open.*

Definition 256 *A point p is a **limit point** of a set C if every open ball centered at p contains a point $q \neq p$ such that $q \in C$.*

Definition 257 *Set C is called **closed** if it contains all its limit points.*

Lemma 258 *A set C is closed if and only if its complement is open.*
Proof. *First, suppose C^c is open. Let x be a limit point of C. Then every open ball centered at x contains a point of C, so that x is not an interior point of C^c. Since C^c is open, this means that $x \in C$. It follows that C is closed.*

Second, suppose that C is closed. Choose $x \in C^c$. Then $x \notin C$, and x is not a limit point of C. Hence there exists an open ball $B_r(x)$ such that $C \cap B_r(x)$ is empty, which implies $B_r(x) \subset C^c$. Thus, x is an interior point of C^c; being true for all $x \in C^c$, this means that C^c is open. ■

Example 259 *A closed ball $B_r(x_0) = \{x \in \mathbb{R}^n | d(x, x_0) \leq r\}$ is a closed set.*

Definition 260 *If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the **closure** of E is the set $E \cup E'$.*

Exercise 261 *Show that empty set $\emptyset$ and the entire space $\mathbb{R}^n$ are both open and closed. Persuade yourself that these two are the only sets which are both open and closed.*

Definition 262 *A set in $\mathbb{R}^n$ is called **compact** if it is closed and bounded.*

This is not the traditional definition of compactness that you will find in a textbook – in spaces more general than $\mathbb{R}^n$ it will not work (that is, in those spaces there exist closed and bounded sets which will not be compact). However, in $\mathbb{R}^n$ it will work fine: whatever definition of compactness you will ever see, it will be equivalent to the one above.

15.5 Convexity and Separating Hyperplanes

There is a branch of real analysis which plays a relatively modest role in pure mathematics, but is an enormously powerful device in economics. It has to do with the notion of convexity.

Unlike topological concepts such as open, closed and compact sets (which in principal require very little structure on the space), convexity makes use of a linear structure.

Definition 263 *A convex combination of points x and y in $\mathbb{R}^n$ is any point z that can be expressed as $z = \alpha x + (1 - \alpha)y$ for some real number $\alpha \in [0, 1]$.*

The set of all convex combinations of two given points is the closed segment between them.

Definition 264 *A set $A \subset \mathbb{R}^n$ is called convex if, together with any two points $x, y \in A$ it contains all their convex combinations.*

Exercise 265 *Show that an intersection of (even infinitely many) convex sets is convex.*

Definition 266 *The convex hull, denoted* conv(A), *of set A is the intersection of all convex sets that contain A. It is the smallest convex set containing A*[10]

Example 267 *An open (or closed) ball is a convex set.*

Example 268 *The half-space is a convex set.*

Definition 269 *Let $p \neq 0$ be a vector in $\mathbb{R}^n$, and let $a \in \mathbb{R}$. The set H defined by $H = \{x \in \mathbb{R}^n | p \cdot x = a\}$ is called a hyperplane in $\mathbb{R}^n$. We denote it by $H(p, a)$.*

Hyperplanes in $\mathbb{R}^2$ are straight lines, hyperplanes in $\mathbb{R}^3$ are usual planes and, generally, hyperplanes in $\mathbb{R}^n$ are spaces of dimension $n - 1$.

The key result (which is indispensable for the second welfare theorem and a variety of other economic results) is the following

Theorem 270 *(Separating Hyperplane Theorem) Let C be a nonempty convex set in $\mathbb{R}^n$ and let x^* be a point in $\mathbb{R}^n$ that is not in C. Then there exists a hyperplane $H(p, a)$ that separates C and x^*, i.e., such that $p \cdot y \leq a$ for all $y \in C$ and $p \cdot x^* \geq a$.*

Exercise 271 *Nonstrict inequalities ($\leq$ and $\geq$) are essential and can not, in general, be replaced by strict inequalities ($<$ and $>$). Construct an example of a convex set and a point outside it that can not be strictly separated.*

A slightly more general result is

Theorem 272 *Let C_1 and C_2 be two disjoint (i.e., $C_1 \cap C_2 = \emptyset$) convex sets in $\mathbb{R}^n$. Then there exists a hyperplane $H(p, a)$ that separates C_1 and C_2, i.e., such that $\forall x \in C_1 \; p \cdot x \leq a$ and $\forall y \in C_2 \;\; p \cdot y \geq a$.*

Example 273 *Any point on the contract curve in the Edgeworth box is a Walrasian equilibrium with an appropriate price vector, as soon as preferences are concave.*

Example 274 *The optimal (from the central planning standpoint) production/consumption choice in a Robinson Crusoe economy can be supported as a decentralized equilibrium, as long as the production possibility set is convex and preferences are concave.*

Finally, here is alternative definition of convex/concave functions whose formulation is closer to the way we defined quasiconvexity/concavity:

Definition 275 *Let $C \subset \mathbb{R}^n$ be a convex set. A function $f : C \to \mathbb{R}$ is called convex, if its epigraph epi$(f) = \{(x, y) \in C \times \mathbb{R} | f(x) \leq y\}$ is a convex set. A function $g : C \to \mathbb{R}$ is concave if its subgraph sub$(f) = \{(x, y) \in C \times \mathbb{R} | f(x) \geq y\}$ is a convex set.*

[10] Likewise, since the intersection of any number of closed sets is closed, we can define the *closure* of set A as the intersection of all closed sets containing A, which will then be the smallest closed superset of A. However, it is straightforward to see that, in general, there will be no such thing as the smallest *open* set containg A (think, for example, of $A = \{0\}$).