

Handout on Level Curves and Their Tangents

Based on Simon and Blume, Section 15.2

10/15/04

Recall the geometric interpretation of the Implicit Function Theorem: a level curve $F(x, y) = c$ defines a curve in the plane, and the theorem gives conditions under which we can think of this curve as defining y as a function of x . The following restatement of the theorem formalizes this intuition:

Theorem 1 *Let (x_0, y_0) be a point on the locus of points $F(x, y) = c$ in the plane, where F is a C^1 function of two variables. If $(\partial F/\partial y)(x_0, y_0) \neq 0$, then $F(x, y) = c$ defines a smooth curve around (x_0, y_0) which can be thought of as the graph of a C^1 function $y = f(x)$. Furthermore, the slope of this curve is*

$$-\frac{\frac{\partial F}{\partial x}(x_0, y_0)}{\frac{\partial F}{\partial y}(x_0, y_0)}. \quad (1)$$

If $(\partial F/\partial y)(x_0, y_0) = 0$, but $(\partial F/\partial x)(x_0, y_0) \neq 0$, then the Implicit Function Theorem tells us that the locus of points $F(x, y) = c$ is a smooth curve about (x_0, y_0) which we can consider as defining x as a function of y . It also tells us that the tangent line to the curve at (x_0, y_0) is parallel to the y -axis, i.e., vertical.

If either $(\partial F/\partial y)(x_0, y_0) \neq 0$ or $(\partial F/\partial x)(x_0, y_0) \neq 0$ holds, we call (x_0, y_0) a **regular point** of the function $F(x, y)$; the theorem tells us that if every point on a particular level set is regular, then that level set defines y as a function of x (or x as a function of y) everywhere on the curve, and that there is a well-defined tangent line to each point on the curve.

(Digression: think about what it would mean if neither of these conditions held. It would mean that from a particular point, a small change in x or y would not change the value of F ; that is, you wouldn't leave the level set. In this case, the level set, at least at this point, is not a curve in the plane at all!)

One application of the theorem is an alternative proof for the theorem of Lagrange. Previously we argued that the Lagrangian method is rooted in the fact that at an optimum, the gradient vectors of the level sets of the objective function and of the constraint function must be parallel. We can make essentially the same argument by observing that because these two level sets must be tangent at an optimum (why?), their

slopes must be equal. If the objective function is F and the equality constraint is G , this implies

$$-\frac{\frac{\partial F}{\partial x}(x^*, y^*)}{\frac{\partial F}{\partial y}(x^*, y^*)} = -\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)} = \lambda \quad (2)$$

which can be rearranged to give

$$\frac{\partial F}{\partial x}(x^*, y^*) + \frac{\partial F}{\partial y}(x^*, y^*) = \lambda\left(\frac{\partial G}{\partial x}(x^*, y^*) + \frac{\partial G}{\partial y}(x^*, y^*)\right), \quad (3)$$

and the theorem of Lagrange follows from this intuition (with a few more details filled in to cover more than two dimensions, multiple constraints - if you're interested, Simon and Blume 19.6 gives the full proof, which relies heavily on the Implicit Function Theorem; it can be argued that the IFT's primary importance is not so much as an optimization tool, but as a foundational lemma for the proofs of the tools that we do use).

Notice, by the way, that this argument gives us another piece of intuition for the rank condition in the theorem of Lagrange (that the gradient vectors of the constraints need to be linearly independent at an optimum for the Lagrange method to work). We can only use the argument above if (x^*, y^*) is a regular point; otherwise (2) is not well-defined. Previously we said that a regular point is one at which a well-defined tangent exists, and we clearly need this to be the case to set up the Lagrangian method as we have. But now, in addition, we note that at a non-regular point, the gradient of $G = [0, 0]$, a failure of the rank condition. Unfortunately, this intuition isn't quite robust to multiple constraints, for the reason we noted in class - with multiple constraints the level sets need not be tangent.