Handout on Second Order Conditions
10/12/04

Theorem 1 *(Second Order Conditions) Suppose* f *is a* C^2 *function on* $Z \subset \mathbb{R}^n$*, and* x^* *is an interior point of* Z*. If* f *has a local maximum (respectively, minimum) at* x^**, then* $D_{x^*}(f)$ *is zero and* $H_f(x^*)$ *is negative (respectively, positive) semidefinite. Conversely, if* $D_{x^*}(f)$ *is zero and* $H_f(x^*)$ *is negative (respectively, positive) definite, then* f *has a strict local maximum (respectively, minimum) at* x^**.*

The above theorem gives almost necessary and sufficient conditions for an interior optimum. Almost – because no conclusions can be drawn if the Hessian is semidefinite but not definite. Besides working out quadratic forms, there is another simple algorithm for testing the definiteness of a symmetric matrix like the Hessian. First, we need some definitions:

Definition 2 *Let* A *be an* $n \times n$ *matrix. A* $k \times k$ *submatrix of* A *formed by deleting* $n-k$ *rows of* A*, and the same* $n-k$ *columns of* A*, is called* ***principal submatrix*** *of* A*. The determinant of a principal submatrix of* A *is called a* ***principal minor*** *of* A*.*

Note that the definition does not specify which $n-k$ rows and columns to delete, only that their indices must be the same.

Example 3 *For a general* 3×3 *matrix,*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

there is one third order principal minor, namely $|A|$*. There are three second order principal minors:*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \text{ formed by deleting column 3 and row 3;}$$

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \text{ formed by deleting column 2 and row 2;}$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \text{ formed by deleting column 1 and row 1}$$

And there are three first order principal minors:

$|a_{11}|$ *, formed by deleting the last two rows and columns*
$|a_{22}|$ *, formed by deleting the first and third rows and columns*
$|a_{33}|$ *, formed by deleting the first two rows and columns*

Definition 4 *Let A by an $n \times n$ matrix. The kth order principal submatrix of A obtained by deleting the last $n - k$ rows and columns of A is called the kth order* ***leading principal submatrix*** *of A, and its determinant is called the kth order* ***leading principal minor*** *of A.*

We will denoted the kth order leading principal submatrix of A by A_k, and its kth order leading principal minor by $|A_k|$. Now, the algorithm for testing the definiteness of a symmetric matrix:

Theorem 5 *Let A be an $n \times n$ symmetric matrix. Then,*

1. (a) *A is positive definite if and only if all its n leading principal minors are (strictly) positive.*
   (b) *A is negative definite if and only if its n leading princiapl minors alternate in sign as follows:*

$$|A_1| < 0,\ |A_2| > 0,\ |A_3| < 0,\ etc.$$

   (c) *If some kth order leading principal minor of A is nonzero but does not fit either of the above sign patterns, then A is indefinite.*

One particular failure of this algorithm occurs when some leading principal minor is zero, but the others fit one of the patterns above. In this case, the matrix is not definite, but may or may not be semidefinite. In this case, we must unfortunately check not only the principal leading minors, but *every* principal minor.

Theorem 6 *Let A be an $n \times n$ symmetric matrix. Then, A is positive semidefinite if and only if every principal minor of A is ≥ 0. A is negative semidefinite if and only if every principal minor of odd order is ≤ 0 and every principal minor of even order is ≥ 0.*