Handout on the Solow Model
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The Solow model is one of the basic building blocks of the Neoclassical Growth Model. Abstracting away from decentralized markets and questions of optimization, the dictatorial model presented here provides insights into how an economy grows given its choice between consumption and saving.

1 The Model

The model uses discrete time (period $t = 0, 1, 2, ...$), and the economy is closed. There is one good produced, which can be consumed or invested into the capital stock. There are two factors of production, capital and labor, but we will be focusing on the role of capital by assuming that labor is inelastically supplied (thus, no questions of wage determination or the choice between labor and leisure will arise). We have the following notation:

- L_t = labor supply at time t
- K_t = capital supply at time t
- Y_t = output at time t
- C_t = consumption at time t
- I_t = investment at time t
- s = the savings rate, exogenously determined by the social planner. Each period, a fraction s of output is invested, and the rest is consumed, disappearing from the economy.
- Lowercase letters will refer to per capita measures. Thus, we have $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $c_t = C_t/L_t$, $i_t = I_t/L_t$.

Output is generated as a function of capital and labor:

$$Y_t = F(K_t, L_t),\ F : \mathbb{R}_+^2 \to \mathbb{R}_+,\ F \text{ is continuous and twice differentiable}$$

Note that output depends on t only through K_t and L_t; F itself is not a function of time. We will assume that F is *neoclassical*, that is, that it satisfies the following assumptions:

1. $F(\mu K, \mu L) = \mu F(K, L)$ (F is homogenous of degree one, or exhibits constant returns to scale)

2. $F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0$

3. $\lim_{K\to 0} F_k = \lim_{L\to 0} F_L = \infty$; $\lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0$

Note that CRS implies, by Euler's Theorem, that

1. $Y = F_K K + F_L L$ (product exhaustion), or, dividing through by output,

2. $1 = \varepsilon_K + \varepsilon_L$ (the elasticities of output with respect to each factor of production sum to one).

Example 1 *The Cobb-Douglass production function* $F(K, L) = K^\alpha L^{1-\alpha}$ *is neoclassical. Note that* $\varepsilon_K = \alpha$ *and* $\varepsilon_L = 1 - \alpha$.

Note also that the assumptions we have made imply that capital and labor are complements:

$$\begin{aligned} F(K, L) &= F_K K + F_L L;\ \text{taking the derivative with respect to } L, \\ F_L &= F_{KL} K + F_{LL} L + F_L \\ -F_{LL} L &= F_{KL} K \end{aligned}$$

which implies that $F_{KL} > 0$.

We will find it convenient to deal in per-capita terms, to focus more explicitly on the role of capital in the economy. To this end, we define

$$y = \frac{1}{L} F(K, L) = F(k, 1) \equiv f(k)$$

We will need to refer to the derivatives of F in terms of f. We have:

$$\begin{aligned} F_K(K, L) &= F_K(k, 1) \equiv f'(k) \text{ because } F_K \text{ is homogenous of degree zero} \\ F_L(K, L) &= \frac{\partial}{\partial L} L f(\frac{K}{L}) \\ &= f(k) - \frac{K}{L^2} f'(k) L \\ &= f(k) - k f'(k) \end{aligned}$$

By the definition of F and f, we have the following other properties of f:

- $f(0) = 0$
- $f'(k) > 0 > f''(k)$
- $\lim_{k\to\infty} f'(k) = 0, \lim_{k\to 0} f'(k) = \infty$

2 Dynamics

Our aim is to look at how the economy changes over time in this model. We begin by simply writing out our assumptions about how things move.

1. The Resource Constraint: $c_t + i_t \leq y_t$

   Note that this says only what's happening to today's output. If there is an existing stock of capital, then it is possible to consume today more than is produced today by eating into the capital stock; this would be negative investment.

2. Population growth: the population grows at the rate $n \geq 0$

   Thus, $L_t = (1+n)L_{t-1} = (1+n)^t L_0$, where L_0 is the initial populaiton. We normalize this to one, so that $L_t = (1+n)^t$.

3. The law of motion for capital: In aggregate terms, $K_{t+1} = (1-\delta)K_t + I_t$, where δ is the rate of physical depreciation. Dividing through by L_t (and noting that $L_t = \frac{L_{t+1}}{(1+n)}$), we can express this in per capita terms as

   $$(1+n)k_{t+1} = (1-\delta)k_t + i_t$$

   When $k_t \approx k_{t+1}$, i.e. near a 'steady state' (which is where we'll be focusing), this can be approximated by

   $$k_{t+1} \approx (1-\delta-n)k_t + i_t$$

   from which we see that we can think of $\delta + n$ as an 'effective rate of depreciation' which takes into account not only the effect of physical depreciation but also of population growth on the per-capita capital stock.

We want to describe the dynamics of the economy in terms of capital and consumption, and ultimately, because consumption is pinned down for us by the social planner, as a function of capital alone. To this end, we combine the resource constraint and the law of motion for capital to get

$$k_{t+1} - k_t = f(k_t) - (\delta+n)k_t - c_t \tag{1}$$

$$c_t = (1-s)f(k_t) \tag{2}$$

The second equation simply links consumption and capital via the savings rate. Combining the two, we get the fundamental equation of the Solow model:

$$k_{t+1} - k_t = sf(k_t) - (\delta+n)k_t \tag{3}$$

This simply says that the change in capital is equal to what is saved out of current output, less depreciation to the current capital stock.

We will find it convenient in examining dynamics to rewrite this equation as an expression of the growth rate of capital. Define $\gamma(k_t) = \frac{k_{t+1}-k_t}{k_t}$ and $\phi(k) = \frac{f(k)}{k}$. Then we can rewrite (3) as

$$\gamma(k) = s\phi(k) - (\delta + n) \tag{4}$$

Note that this equation is stationary; $\gamma(\cdot)$ depends on time only through the subscript on k_t; here we are expressing the dynamics of the Solow model in terms of the capital stock only, as well as the parameters of the model, irrespective of the time period.

3 Steady State

We are interested in:

- whether or not the economy ever comes to a steady state - defined as any k^* such that if $k_s = k^*$, $k_t = k^* \forall t \geq s$
- if so, how the economy arrives there, and
- whether the steady state is stable, in the sense that if we begin at steady state and then perturb the economy slightly, the economy will return to the steady state.

One steady state is the trivial one at which $c = k = 0$. We are more interested in some non-zero steady state, at which, by definition,

$$\begin{aligned} \gamma(k^*) &= s\phi(k^*) - (\delta + n) = 0, \text{ or, equivalently,} \\ \phi(k^*) &= \frac{\delta + n}{s} \end{aligned} \tag{5}$$

Let's look at this second equation more carefully (drawing pictures will help). Consider the function $\phi(k)$. Its definition and the properties of f tell us that it is continuous and twice differentiable. Moreover, we have

$$\phi'(k) = \frac{\partial}{\partial k}\frac{f(k)}{k} = \frac{f'(k)}{k} - \frac{f(k)}{k^2} = \frac{kf'(k) - f(k)}{k^2} = -\frac{F_L}{k^2} < 0$$

so that we know that $\phi(k)$ is decreasing. Additionally, by L'Hopital's rule we know that $\phi(0) = f'(0) = \infty$ and $\phi(\infty) = f'(\infty) = 0$. We therefore have a decreasing function mapping the positive reals onto the positive reals, and the (5) has a unique solution iff $\frac{\delta+n}{s}$ is positive. This gives us our steady state:

$$\phi(k^*) = \frac{\delta+n}{s}, \text{ so}$$
$$k^* = \phi^{-1}(\frac{\delta+n}{s}), \ \phi^{-1} \text{ also decreasing}$$

Note that:

- $\gamma(k^*) = 0$; $\phi(k)$ decreasing tells us that $\gamma(k) > 0$ for $k < k^*$ and $\gamma(k) > 0$ for $k > k^*$. Thus, the steady state is not just locally but *globally* stable.

- ϕ^{-1} decreasing tells us that k^* is decreasing in δ and n and increasing in s

- However, $c^* = (1-s)f(k^*)$, so while c^* is unambiguously decreasing in δ and n, s has an ambiguous effect on steady-state consumption (the relationship turns out to be quadratic).

- $\phi(k)$ decreasing also tells us that $\gamma(k)$ is monotonically decreasing in k. Thus, when k is below steady state the economy grows at a rate which is decreasing to the steady state, and conversely when k is above steady state the economy contracts at a rate which is decreasing to the steady state.

Let us prove more precisely that there is a stable steady state, that the economy will converge to it from any initial level of capital, and that it will do so at a monotonically decreasing rate.

To this end, define $G(k) = sf(k) + (1-\delta-n)k$; note that $G(k)$ represents 'tomorrow's' capital stock given that capital 'today' is equal to k. Assuming that the effective rate of depreciation is less than one, we have

$$G'(k) = sf'(k) + 1 - \delta - n > 0$$
$$G''(k) = sf''(k) < 0$$

So G is increasing and concave. Moreover,

$$G(0) = 0, G'(0) = \infty$$
$$G(\infty) = \infty, G'(\infty) = 1 - \delta - n < 1$$
$$G(k^*) = k^*$$

Again, a picture helps with intuition. We have an increasing and concave function which begins at the origin and whose slope asymptotes to something positive but less than one. This tells us that $G(k) > k$ for all $k < k^*$ and $G(k) < k$ for all $k > k^*$; i.e., the fixed point $G(k^*) = k^*$ is the unique fixed point (other than the origin itself).

Now, consider a sequence $\{k_t\}_{t=0}^{\infty}, k_0 \in (0, k^*)$. We know that in this sequence, for every t, $k_t < k_{t+1} < k^*$, that is, the sequence is monotonically increasing and bounded above by k^*. It must, therefore, converge to some $\widehat{k} \leq k^*$. But the fact that $G(k)$ is continuous means that $G(\widehat{k}) = \widehat{k}$ (more precisely, $G(\widehat{k})$ can be made arbitrarily close to $\widehat{k}$). That is, $\widehat{k}$ is a fixed point of G. But we just saw that k^* is the unique fixed point of G; thus, $\widehat{k} = k^*$. We have thus shown that for any initial $k_0 \in (0, k^*)$, the sequence $\{k_t\}_{t=0}^{\infty}$ will converge to k^*. A symmetric argument shows that for any $k_0 \in (k^*, \infty)$, the sequence $\{k_t\}_{t=0}^{\infty}$ will converge to k^* from above. This shows that the steady state is indeed stable and that the economy will converge to it from any initial level of capital.

To see that the rate at which it does so is monotonically decreasing at the steady state is approached, note the following:

$$\begin{aligned}
\gamma(k) &= s\phi(k) - (\delta + n) \\
\gamma(k^*) &= 0 \\
\gamma(k) &> 0 \text{ if } k < k^* \text{ and } \gamma(k) < 0 \text{ if } k > k^* \\
\gamma'(k) &= s\phi'(k) < 0
\end{aligned}$$

Putting these together gives us

$$\begin{aligned}
\gamma(k_t) &> \gamma(k_{t+1}) > \gamma(k^*) = 0 \text{ if } k_t \in (0, k^*) \\
\gamma(k_t) &< \gamma(k_{t+1}) < \gamma(k^*) = 0 \text{ if } k_t \in (k^*, \infty)
\end{aligned}$$

which is what we wanted to prove: capital grows to the steady state at a decreasingly positive rate from below, and contracts at a decreasingly negative rate from above.

4 Continuous Time Dynamics with Log-Linearization

Finally, we can see the same aspects of the Solow model in continuous time using a technique called log-linearization. To do this, we first rewrite the dynamics of the Solow model in continuous time:

$$\gamma(k) = \frac{(dk/dt)}{k} \equiv \frac{\dot{k}}{k} = s\phi(k) - (\delta + n) \tag{6}$$

Now, we define a new variable $z = \log k - \log k^*$. Note that this is a 'function' of k, with k^* being a constant, and that $z = 0$ precisely when $k = k^*$. So we have

$$k = k^* e^z \tag{7}$$

and

$$\dot{k} = \frac{dk}{dt} = \frac{dk^* e^z}{dt} = k^* e^z \frac{dz}{dt} = k\dot{z} \tag{8}$$

or

$$\dot{z} = \frac{\dot{k}}{k} \tag{9}$$

Moreover, we can express $\gamma(k)$ in terms of z :

$$s\phi(k) - (\delta + n) = s\phi(k^* e^z) - (\delta + n) \equiv \Gamma(z) \tag{10}$$

So that combining (6), (9), and (10), we have

$$\dot{z} = \Gamma(z) \tag{11}$$

Now, as we said, $z = 0$ precisely when $k = k^*$. We can therefore use a linear approximation of (11) about 0 to understand the dynamics of the system in z, and because this approximation will be about a steady state it will be quite accurate (i.e., neither the definition of z nor the choice to take a Taylor expansion about 0 is arbitrary). This approximation will in turn help us to understand the dynamics of the system in k, which is our main interest.

First, let us understand a bit more the function $\Gamma(z)$. We have:

$$\Gamma(z) \text{ defined for all } z \in \mathbb{R} \tag{12}$$

$$\Gamma(0) = s\phi(k^*) - (\delta + n) = 0 \tag{13}$$

$$\Gamma(z) > 0 \forall z < 0, \Gamma(z) > 0 \forall z > 0 \tag{14}$$

$$\Gamma'(z) = s\phi'(k^* e^z) k^* e^z \tag{15}$$

and in this last equation it is helpful to note that

$$\phi'(k) = \frac{kf'(k) - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right]\frac{f(k)}{k^2} = -(1 - \varepsilon_k)\frac{f(k)}{k^2} \tag{16}$$

Now we are ready to take a Taylor expansion of $\Gamma(z)$ about 0 :

$$\Gamma(z) = \Gamma(0) + \Gamma'(0)z \tag{17}$$

$$= \Gamma'(0)z \text{ (using (13))} \tag{18}$$

which, using (15) and (16), gives us

$$\Gamma(z) = s\phi'(k^*)k^* z \tag{19}$$

$$= -s(1 - \varepsilon_k)\frac{f(k^*)}{k^{*2}} k^* z \tag{20}$$

$$= -s\frac{f(k^*)}{k^*}(1 - \varepsilon_k)z \tag{21}$$

$$= -(\delta + n)(1 - \varepsilon_k)z \text{ (using (13) once again)} \tag{22}$$

$$\equiv \lambda z, \text{ where } \lambda < 0 \tag{23}$$

Recalling that $z = \log k - \log k^*$ and $\dot{z} = \frac{\dot{k}}{k}$, this gives us the system in k:

$$\frac{\dot{k}}{k} = \lambda \log(\frac{k}{k^*}),\ \lambda < 0 \tag{24}$$

which simply tells us what we had already learned from the discrete time model. The growth rate of capital is monotonically decreasing in the capital stock; it is 0 when $k = k^*$. Again, this tells us that the system has a stable steady state at k^*.