14.102 Problem Set 1

Due Thursday, September 23, 2004, in class

1. Let $A = \begin{pmatrix} 4 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 1 \end{pmatrix}$

   (a) Find $C = AB$

   (b) Find $rank\ C$

   (c) Find $\det C$

   (d) Find $D = BA$

   (e) Find $rank\ D$

   (f) Find $\det D$

   (g) Is C invertible? If so, find C^{-1}

   (h) Is D invertible? If so, find D^{-1}

   (i) Find eigenvalues of C

   (j) Solve the following two linear systems (Hint: you will need no extra calculations!):

      i. $\begin{cases} 3x + 2y &= 1 \\ 5x + 3y &= 0 \end{cases}$

      ii. $\begin{cases} 3u + 2v &= 0 \\ 5u + 3v &= 1 \end{cases}$

2. Lecture Notes Exercise 13: Given an $m \times n$ matrix A, show that $S(B) \subseteq S(A)$ and $N(A') \subseteq N(B')$ whenever $B = AX$ for some matrix X. What is the geometric interpretation?

3. Lecture Notes Exercise 19/Lemma 20: Suppose $\{e_j\}$ is a basis for $\mathbb{X}$; let $P = [p_{ij}]$ be any nonsingular $n \times n$ matrix, and let $f_j = \sum_i p_{ij} e_i$. Show then that $\{f_j\}$ is a basis for $\mathbb{X}$ as well.

4. For a square matrix A assume that all elements of both A and A^{-1} are integers. What values can $\det A$ take?

5. Lecture Notes Exercise 36: Using the properties of transpose and inverse:

   (a) Prove that $A^{-k} = (A^k)^{-1}$

   (b) Consider the matrix $Z = X(X'X)^{-1}X'$ where X is an arbitrary $m \times n$ matrix. Under what conditions on X is Z well-defined? Show that Z is symmetric. Also show that $ZZ = Z$ (i.e., that Z is **idempotent**).

6. Lecture Notes Exercise 42: Show that for a 2×2 matrix

$$A = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix}$$

provided $|A| = \alpha\delta - \beta\gamma \neq 0$, the inverse is

$$A^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

7. Lecture Notes Exercise 49: Prove the claim made in the lecture notes that if we can find as many as k linearly independent solutions $x^1, ..., x^k$ to $Ax = 0$, then any $z \in S[x^1, ..., x^k]$ is a solution as well. That is, prove that $Ax^1 = Ax^2 = 0 \Rightarrow Az = 0 \, \forall z \in S[x^1, x^2]$.

8. Lecture Notes Exercise 55: Show that $null[A, b] = 1$ if and only if $rank[A, b] = rank(A)$.

9. (Harder – for extra credit). Let

$$d_n = \det \begin{pmatrix} 1 & 1 & 0 & 0 & ... & ... & 0 \\ -1 & 1 & 1 & 0 & ... & ... & 0 \\ 0 & -1 & 1 & 1 & ... & ... & 0 \\ 0 & 0 & -1 & 1 & ... & ... & 0 \\ 0 & 0 & 0 & -1 & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & 1 \\ 0 & 0 & 0 & 0 & ... & -1 & 1 \end{pmatrix}$$

This is an $n \times n$ matrix with ones on and right above the diagonal, negative ones right below the diagonal and zeros elsewhere.

Show that d_n is equal to the $(n+1)$th term of the Fibonacci sequence.