14.102 Problem Set 2

Due Thursday, October 7, 2004, in class

1. Lecture Notes Exercise 78: Consider the 2×2 identity matrix. What are its eigenvalues? Find a $V = [v_1 \ v_2]$ such that $V'V = I$ and $V^{-1}IV = I$. What are the corresponding $\{\mathbb{M}_1, \mathbb{M}_2\}$? Consider now

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

   Find an orthonormal V and a diagonal Λ such that $V'AV = \Lambda$. Hint: remember that if v is an eigenvector, then ϕv is also an eigenvector for any scalar ϕ.

2. Lecture Notes Exercise 91: Show that if X is symmetric and idempotent, then X is also positive semi-definite. Note that prior to 9/29, 'and idempotent' was missing from the lecture notes, but is needed! Optional: can you see why?

3. Give an example of a function that is continuous at exactly one point (say, 0) and is also differentiable at this point.

4. Lecture Notes Exercise 122: Find the domains of the following functions $f : \mathbb{R} \to \mathbb{R}$:

   (a) $f(x) = \sqrt{x}$ (note: we typically adopt the convention that the square root of x refers to the positive root, unless explicitly stated otherwise)

   (b) $f(x) = \frac{1}{x^2+2x-3}$

   (c) $f(x) = \frac{1}{\sin x} + \frac{1}{\cos x}$

5. (Sundaram 4.4, page 110) For each of the following functions, state whether the conditions of the Weierstraß theorem apply. Find and classify all critical points (local maximum, local minimum, neither) of each of the following functions. For local optima that you find figure out whether they are also global optima. Try to save your time by avoiding using second order approach wherever possible.

   (a) $f(x, y) = x \sin y$

   (b) $f(x, y) = \frac{1}{x} + \frac{1}{y}$

   (c) $f(x, y) = x^4 + y^4 - x^3$

6. (Simon and Blume 15.6, page 342) Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

   (a) If $x_1 = 6$ and $x_2 = 3$, find a y which satisfies $F(x_1, x_2, y) = 0$.

(b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1 = 6$, $x_2 = 3$?

(c) If so, compute $(\frac{\partial y}{\partial x_1})(6, 3)$ and $(\frac{\partial y}{\partial x_2})(6, 3)$.

(d) If x_1 increases to 6.2 and x_2 decreses to 2.9, estimate the corresponding change to y.