14.102 Problem Set 3

Due Thursday, October 21, 2004, in class

Starred (*) problems will not count for the grade on this problem set; they are based on material from lectures on 10/21 and 10/26, and provide practice for the midterm on 10/28. If you would like to do them prior to 10/21 and hand them, I will be happy to make comments. Solutions to all problems will be posted after class on 10/21.

1. Lecture Notes Exercise 168: For a function f defined on a convex subset U in $\mathbb{R}^n$, show that f concave implies f quasiconcave.

2. (Sundaram 5.11, page 144) Consider the problem of maximizing the utility function $u(x,y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$ on the budget set $px + y = 1, x \geq 0, y \geq 0$. Show that if non-negativity constrains are ignored, and the problem is written as an equality-constrained one, the resulting Lagrangean has a unique critical point. Does this critical point identify a solution to the problem? Why or why not?

3. (Sundaram 6.12, page 171) A firm produces a single output y using three inputs x_1, x_2, x_3 in nonnegative quantities through the relationship $y = x_1(x_2 + x_3)$.The unit price of y is $p_y > 0$ while that of the input x_i is $w_i > 0, i = 1, 2, 3$.

   (a) Describe the firm's profit-maximization problem and derive the equations that define the critical points of the Lagrangean L in this problem.

   (b) Show that the Lagrangean L has multiple critical points for any choice of $(p_y, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$.

   (c) Show that none of these critical points identifies a solution of the profit-maximization problem. Can you explain why this is the case?

4. (Sundaram 8.25, page 201) An agent who consumes three commodities has a utility function given by $u(x_1, x_2, x_3) = \sqrt[3]{x_1} + \min\{x_2, x_3\}$. Given an income of I and prices p_1, p_2, p_3, write down the consumer's utility-maximization problem (you need not solve it[1]). Can the Weierstraß and/or Kuhn-Tucker theorems be used to obtain and characterize a solution (that is, are they applicable to this problem)? Why or why not?

5. Lecture Notes Exercise 205: Compute $\int_a^{+\infty} te^{-rt}dt$ (use integration by parts).

6. Lecture Notes Exercise 206: Compute $\int_0^{+\infty} e^{-\sqrt{t}}dt$ (use the change of variable $u = \sqrt{t}$).

[1] If you ever take a class from Bengt, or an exam he writes, you will become very used to problems that require you only to set up the program (but which can still be surprisingly difficult!).

7. Lecture Notes Exercise 211: For each of the following relations, show that R is (or is not) reflexive, symmetric, and transitive. In each, $x, y \in \mathbb{R}^n$.

   (a) xRy if $x_1 > y_1$, where x_1 and y_1 are the respective first elements of x and y.

   (b) xRy if $x_1 = y_1$

   (c) xRy if $\|x\| = \|y\|$

8. Let A be a set of nonempty real numbers which is bounded below. Let $-A$ be the set of all real numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

9. (*) Prove carefully that the sum of two convergent sequences is convergent and its limit is the sum of the limits.

10. (*) Find all limit points of the following sequence: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4,...

11. (*) Show that the intersection of (even infinitely many) closed sets is closed. Give an example of an infinite family of closed sets whose union is not closed.

12. (*) Let $A = [-1; 0)$ and $B = (0, 1]$. Examine whether each of the following statements is true or false:

   (a) $A \cup B$ is compact;

   (b) $A + B = \{x + y | x \in A, y \in B\}$ is compact;

   (c) $A \cap B$ is compact.

13. (*) Define

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find an open set O such that $f^{-1}(O)$ is not open and a closed set C such that $f^{-1}(C)$ is not closed.

14. (*) (Harder) Start from any set $A \in \mathbb{R}^n$. Consider the following two operations: taking closure of a set and taking convex hull of a set. At most how many *distinct* sets can one obtain by consecutively applying these operations to A (in any order)? Try to show that the number you get is indeed the maximum.