14.102 Problem Set 1 Solutions

1. Let $A = \begin{pmatrix} 4 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 1 \end{pmatrix}$

   (a) Find $C = AB$

   Solution: $C = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}$

   (b) Find *rank* C

   Solution: 2

   (c) Find det C

   Solution: -1

   (d) Find $D = BA$

   Solution: $D = \begin{pmatrix} 10 & 2 & -3 \\ -12 & -3 & 6 \\ 6 & 1 & -1 \end{pmatrix}$

   (e) Find *rank* D

   Solution: Any two columns of D are linearly independent, so $rankD$ is at least 2. On the other hand, it can not be above 2, since rank of the product is no greater than rank of each of the matrices being multiplied (why?). So $rankD = 2$

   (f) Find det D

   Solution: 0, since D is not full rank.

   (g) Is C invertible? If so, find C^{-1}

   Solution: Yes, $C^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$

   (h) Is D invertible? If so, find D^{-1}

   Solution: No, since $\det D = 0$.

   (i) Find eigenvalues of C

   Solution: We have to solve $\begin{vmatrix} 3-\lambda & 2 \\ 5 & 3-\lambda \end{vmatrix} = 0 \Longleftrightarrow (3-\lambda)^2 = 10 \Longleftrightarrow 3-\lambda = \pm\sqrt{10} \Longleftrightarrow \lambda = 3 \pm \sqrt{10}$

   (j) Solve the following two linear systems (Hint: you will need no extra calculations!):

   i. $\begin{cases} 3x + 2y & = & 1 \\ 5x + 3y & = & 0 \end{cases}$

ii. $\begin{cases} 3u+2v &= 0 \\ 5u+3v &= 1 \end{cases}$

Solution: We have $C\begin{pmatrix} x & u \\ y & v \end{pmatrix} = I \iff \begin{pmatrix} x & u \\ y & v \end{pmatrix} = C^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$

2. Lecture Notes Exercise 13: Given an $m \times n$ matrix A, show that $S(B) \subseteq S(A)$ and $N(A') \subseteq N(B')$ whenever $B = AX$ for some matrix X. What is the geometric interpretation?

   Solution: Suppose X is $n \times l$. Then B is $m \times l$. We have $S(A) = \{y \in \mathbb{R}^m | y = Ax$ for some $x \in \mathbb{R}^n\}$, and $S(B) = \{y \in \mathbb{R}^m | y = Bx$ for some $x \in \mathbb{R}^l\}$. We want to show that any $y \in S(B)$ belongs to $S(A)$ as well. We have $y = Bx = AXx = Az$, where $z = Xx, z \in \mathbb{R}^n$, implying that $y \in S(B) \Longrightarrow y \in S(A)$.

   For the second part, recall that $N(A') = \{x \in R^m | A'x = 0\}$, and $N(B') = \{x \in R^m | B'x = 0\}$. We want to show that $x \in N(A') \Rightarrow x \in N(B')$, and the proof is similar to the previous part: if $A'x = 0$, then we have $B'x = X'A'x = 0$.

3. Lecture Notes Exercise 19/Lemma 20: Suppose $\{e_j\}$ is a basis for $\mathbb{X}$; let $P = [p_{ij}]$ be any nonsingular $n \times n$ matrix, and let $f_j = \sum_i p_{ij} e_i$. Show then that $\{f_j\}$ is a basis for $\mathbb{X}$ as well.

   Solution: As noted in the lecture notes, this is equivalent to saying that if the matrix E, with columns consisting of the vectors $\{e_j\}$, is a basis for $\mathbb{X}$, then $F = EP$ is a basis for $\mathbb{X}$ as well. Note that since P is nonsingular, we can write $E = FP^{-1}$. First let us show that F spans $\mathbb{X}$. This is equivalent to saying that any $x \in \mathbb{X}$ can be written as $x = Fc$, where c are the coordinates of x under the basis F. We already know that E is a basis for $\mathbb{X}$. So for any $x \in \mathbb{X}$, we can write $x = Ed = FP^{-1}d = Fc$, as desired. $c = P^{-1}d$, the coordinates of x under the basis F, is simply the product of the inverted projection matrix P^{-1} and d, the coordinates of x under the basis E.

   What is meant by 'projection matrix'? That is, what is the role of matrix P above? Notice that $F = EP$ or $f_j = Ep_j$ means that p_j's are the coordinates of f_j's under the basis E. Thus, if c are the coordinates of x under E, Pc are its coordinates under F. This all means that the transformation $c \mapsto Pc$ for $P = E^{-1}F$ just gives the new coordinates on basis E given initial coordinates under basis F. And indeed P is just the projection of F on E. We used the inverted projection matrix above because we were going the other way: given initial basis E, we were looking for the coordinates of x under the new basis F.

   What remains to be shown is that the columns of F are linearly independent. But this is immediate. One way to see it is that since P is nonsingular, the dimensions of F are the same as the dimensions of E

- so if F spans $\mathbb{X}$, its columns must be linearly independent. Another approach is to note that F is invertible: $F^{-1} = P^{-1}E^{-1}$.

4. For a square matrix A assume that all elements of both A and A^{-1} are integers. What values can $\det A$ take?

   Solution: If all elements of a matrix are integers, then so is its determinant. We have two integers, $\det A$ and $\det(A^{-1})$, whose product is equal to $\det I = 1$. The only two possibilities are $\det A = 1$ and $\det A = -1$.

5. Lecture Notes Exercise 36: Using the properties of transpose and inverse:

   (a) Prove that $A^{-k} = (A^k)^{-1}$
   Solution: $A^{-k} = (A^{-1})^k = A^{-1}A^{-1}\cdots A^{-1}$ (k times) $= (A^k)^{-1}$ by the property that $(AB)^{-1} = B^{-1}A^{-1}$.

   (b) Consider the matrix $Z = X(X'X)^{-1}X'$ where X is an arbitrary $m\times n$ matrix. Under what conditions on X is Z well-defined? Show that Z is symmetric. Also show that $ZZ = Z$ (i.e., that Z is **idempotent**).
   Solution: $(X'X)$ must be invertible. This can only be the case if $n < m$ (why?) and if $rank X = n$.
   To show that Z is symmetric, note that $(X'X)$ is symmetric, and hence so is $(X'X)^{-1}$ (why?). Now $Z' = (X(X'X)^{-1}X')' = X''(X'X)^{-1\prime}X' = X(X'X)^{-1}X' = Z$.
   To show that Z is idempotent, we check that $ZZ = (X(X'X)^{-1}X')(X(X'X)^{-1}X') = X(X'X)^{-1}(X'X)(X'X)^{-1}X' = X(X'X)^{-1}IX' = X(X'X)^{-1}X = Z$.

6. Lecture Notes Exercise 42: Show that for a 2×2 matrix

$$A = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix}$$

   provided $|A| = \alpha\delta - \beta\gamma \neq 0$, the inverse is

$$A^{-1} = \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

   Solution: This follows from the algorithm given in the lecture notes; we can also check that

$$\begin{aligned} AA^{-1} &= \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \\ &= \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \alpha\delta - \beta\gamma & 0 \\ 0 & \alpha\delta - \beta\gamma \end{bmatrix} \\ &= I \\ &= \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}\begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix} \\ &= A^{-1}A \end{aligned}$$

7. Lecture Notes Exercise 49: Prove the claim made in the lecture notes that if we can find as many as k linearly independent solutions $x^1, ..., x^k$ to $Ax = 0$, then any $z \in S[x^1, ..., x^k]$ is a solution as well. That is, prove that $Ax^1 = Ax^2 = 0 \Rightarrow Az = 0 \, \forall z \in S[x^1, x^2]$.

   Solution: If $z \in S[x^1, x^2]$, z can be written as a linear combination of x^1 and x^2, such as $\alpha_1 x_1 + \alpha_2 x_2$. Then $Az = A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2 = 0$.

8. Lecture Notes Exercise 55: Let A have full rank. Show that $null[A, b] = 1$ if and only if $rank[A, b] = rank(A)$.

   Solution: We have that $rank[A, b] = rank(A) = n$. This implies, first, that $[A, b]$ is not full rank (since it has $n+1$ columns). Then b can be written as $b = Ax$. We know, however, that such a square system has a unique solution if and only if A has full rank, which it does, so the vector x which solves $Ax = b$ is unique (up to scalar multiplication). Rewrite this system as the homogenous system $[A, b]y = 0$, where $y = \begin{bmatrix} x \\ -1 \end{bmatrix}$. Because x is unique, so is y; more over, y is nonzero. There is thus exactly one vector solving $[A, b]y = 0$, which is equivalent to $null[A, b] = 1$. Note well that although we went in the (if) direction, every step involved an equivalence - in other words, the same argument in reverse establishes (only if). The only thing we must note is that to move from $[A, b]y = 0$ to $Ab = x$, we must choose the point on the line described by y (i.e., the scalar multiple of y) with last coordinate -1. Also, the last step requires that A being of full rank be a hypothesis maintained throughout.

9. (Harder – for extra credit). Let

$$d_n = \det \begin{pmatrix} 1 & 1 & 0 & 0 & ... & ... & 0 \\ -1 & 1 & 1 & 0 & ... & ... & 0 \\ 0 & -1 & 1 & 1 & ... & ... & 0 \\ 0 & 0 & -1 & 1 & ... & ... & 0 \\ 0 & 0 & 0 & -1 & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & 1 \\ 0 & 0 & 0 & 0 & ... & -1 & 1 \end{pmatrix}$$

This is an $n \times n$ matrix with ones on and right above the diagonal, negative ones right below the diagonal and zeros elsewhere.

Show that d_n is equal to the $(n+1)$th term of the Fibonacci sequence.

Solution: Fibonacci sequence is defined by $a_1 = a_2 = 1$, $a_k = a_{k-1} + a_{k-2}$. We have $d_1 = a_2 = 1$ and $d_2 = a_3 = 2$. All that remains is to show that $d_k = d_{k-1} + d_{k-2}$ for all k. That is done by applying the definition of the determinant as given in class: in the decomposition with respect to the first column only two terms will be nontrivial. The required formula follows immediately.